

Expansion in high dimension for the growth constants of lattice trees and lattice animals

Yuri Mejía Miranda* and Gordon Slade†

August 16, 2012

Abstract

We compute the first three terms of the $1/d$ expansions for the growth constants and one-point functions of nearest-neighbour lattice trees and lattice (bond) animals on the integer lattice \mathbb{Z}^d , with rigorous error estimates. The proof uses the lace expansion, together with a new expansion for the one-point functions based on inclusion-exclusion.

AMS 2010 Subject Classification: 60K35, 82B41

1 Main result

For $d \geq 1$, we consider the integer lattice \mathbb{Z}^d as a regular graph of degree $2d$, with edges consisting of the nearest-neighbour bonds $\{x, y\}$ with $\|x - y\|_1 = 1$. A *lattice animal* is a finite connected subgraph, and a *lattice tree* is a lattice animal without cycles. These are fundamental objects in combinatorics and in the theory of branched polymers [21].

We denote the number of lattice animals containing n bonds and containing the origin of \mathbb{Z}^d by a_n , and the number of lattice trees containing n bonds and containing the origin of \mathbb{Z}^d by t_n . Standard subadditivity arguments [22, 23] provide the existence of the d -dependent *growth constants* (which we express in the notation of [8])

$$\lambda_0 = \lim_{n \rightarrow \infty} t_n^{1/n}, \quad \lambda_b = \lim_{n \rightarrow \infty} a_n^{1/n}. \quad (1.1)$$

A deeper analysis shows that $\lambda_0 = \lim_{n \rightarrow \infty} t_{n+1}/t_n$ and $\lambda_b = \lim_{n \rightarrow \infty} a_{n+1}/a_n$ [24]. The *one-point functions* are the generating functions of the sequences a_n and t_n , namely

$$g^{(t)}(z) = \sum_{n=0}^{\infty} t_n z^n \quad \text{and} \quad g^{(a)}(z) = \sum_{n=0}^{\infty} a_n z^n. \quad (1.2)$$

*Department of Mathematics, University of British Columbia, Vancouver, BC, Canada V6T 1Z2. Email: yuri@math.ubc.ca, slade@math.ubc.ca

†Department of Mathematics, University of British Columbia, Vancouver, BC, Canada V6T 1Z2. Email: slade@math.ubc.ca

These have radii of convergence $z_c^{(t)} = \lambda_0^{-1}$ and $z_c^{(a)} = \lambda_b^{-1}$, respectively. We refer to $z_c^{(t)}$ and $z_c^{(a)}$ as the *critical points*. We use superscripts to differentiate between lattice trees and lattice animals, and we write z_c or $g(z)$ below for statements that apply to both models. We use the abbreviation

$$g_c = g(z_c). \quad (1.3)$$

Also, to make statements simultaneously for lattice trees and lattice animals, we use the indicator function $\mathbb{1}_a$ which takes the value 1 for the case of lattice animals, and the value 0 for the case of lattice trees.

Our main result is the following theorem, which gives detailed information on the asymptotic behaviour of the critical points and critical one-point functions as $d \rightarrow \infty$. The notation $f(d) = o(h(d))$ means $\lim_{d \rightarrow \infty} f(d)/h(d) = 0$.

Theorem 1.1. *For lattice trees or lattice animals, as $d \rightarrow \infty$,*

$$z_c = e^{-1} \left[\frac{1}{2d} + \frac{\frac{3}{2}}{(2d)^2} + \frac{\frac{115}{24} - \mathbb{1}_a \frac{1}{2} e^{-1}}{(2d)^3} \right] + o(2d)^{-3}, \quad (1.4)$$

$$g_c = e \left[1 + \frac{\frac{3}{2}}{2d} + \frac{\frac{263}{24} - \mathbb{1}_a e^{-1}}{(2d)^2} \right] + o(2d)^{-2}. \quad (1.5)$$

Theorem 1.1 extends our results in [26], where it was proved that, for both models,

$$z_c = \frac{1}{2de} + o(2d)^{-1}, \quad g_c = e + o(1). \quad (1.6)$$

These leading terms could be extracted relatively easily from the lace expansion results of [12, 13], together with a comparison with the mean-field model studied in [3]. The lattice trees and lattice animals we are considering are *bond* clusters. For the closely related models of *site* trees and *site* animals, it was proved in [1] and [2] respectively, using very different methods than ours, that the corresponding growth constants Λ_0 and Λ_s (in the notation of [8]) are both asymptotic to $2de$ as $d \rightarrow \infty$. For related results for spread-out models of lattice trees and lattice animals, see [29, 26].

The behaviour of $z_c^{(t)}$ and $z_c^{(a)}$ as $d \rightarrow \infty$ has been extensively studied in the physics literature. For lattice trees, the expansion

$$z_c^{(t)} = e^{-1} \left[\frac{1}{2d} + \frac{\frac{3}{2}}{(2d)^2} + \frac{\frac{115}{24}}{(2d)^3} + \frac{\frac{309}{16}}{(2d)^4} + \frac{\frac{619103}{5760}}{(2d)^5} + \frac{\frac{543967}{768}}{(2d)^6} \right] + \dots \quad (1.7)$$

is equivalent to the expansion given in [8] for λ_0 , but in [8] no rigorous estimate for the error term is obtained. Similarly, the series

$$z_c^{(a)} = e^{-1} \left[\frac{1}{2d} + \frac{\frac{3}{2}}{(2d)^2} + \frac{\frac{115}{24} - \frac{1}{2}e^{-1}}{(2d)^3} + \frac{\frac{309}{16} - 2e^{-1}}{(2d)^4} + \frac{\frac{619103}{5760} - \frac{113}{12}e^{-1}}{(2d)^5} + \frac{\frac{543967}{768} - \frac{395}{12}e^{-1} - \frac{55}{24}e^{-2}}{(2d)^6} \right] + \dots \quad (1.8)$$

is equivalent to the result of [16, 27] for λ_b , but again no rigorous error estimate was obtained in [16, 27]. Equation (1.4) provides rigorous confirmation of the first three terms in (1.7)–(1.8), using completely different methods than [8, 16, 27].

The formulas (1.4)–(1.8) are examples of $1/d$ expansions. Such expansions have a long history and have been developed for several models, in particular for self-avoiding walk and percolation. Let c_n denote the number of n -step self-avoiding walks starting at the origin. For nearest-neighbour self-avoiding walk on \mathbb{Z}^d , it was proved in [15] that the inverse connective constant $z_c^{(s)} = [\lim_{n \rightarrow \infty} c_n^{1/n}]^{-1}$ has an asymptotic expansion $z_c^{(s)} \sim \sum_{i=1}^{\infty} m_i (2d)^{-i}$ to all orders, with all coefficients m_i *integers*. The first six coefficients had been computed much earlier, in [7], but without rigorous control of the error, and these six values were confirmed with rigorous error estimate in [15]. Subsequently, seven additional coefficients in the expansion were computed in [4]. The values of m_i for $i \leq 11$ are positive, whereas m_{12} and m_{13} are negative. It appears likely that the series $\sum_i m_i x^i$ has radius of convergence equal to zero. It may however be Borel summable, and a partial result in this direction is given in [11]. Some related results for nearest-neighbour bond percolation on \mathbb{Z}^d are obtained in [15, 18, 19]. In particular, it is shown in [18] that the critical probability $p_c = p_c(d)$ has an asymptotic expansion $p_c \sim \sum_{i=1}^{\infty} q_i (2d)^{-i}$ to all orders, with all q_i *rational*. The values of q_1, q_2, q_3 are computed in [15, 19], and q_i is given for $i \leq 5$ in [10] but without rigorous error estimate. Results for *spread-out* models of percolation and self-avoiding walk can be found in [17, 28, 29].

An interesting problem which we do not solve in this paper is to prove existence of asymptotic expansions to all orders for $z_c^{(t)}$ and $z_c^{(a)}$; we believe that the methods we develop would be useful for approaching this problem. An existence proof would then open up the additional problems of proving that the series have zero radius of convergence but are Borel summable—the latter problems seem considerably more difficult than the existence problem. Also, both the formula (1.7) and the insights in our proof strongly suggest that there exists an asymptotic expansion $z_c^{(t)} \sim e^{-1} \sum_{i=-1}^{\infty} r_i (2d)^{-i}$, with r_i *rational*, but we do not prove this either. The formula (1.4) does prove that the coefficients for $z_c^{(a)}$ are not all rational multiples of e^{-1} , as was already apparent from the nonrigorous formula (1.8). In our proof, the appearance of the term $-\frac{1}{2}e^{-1}$ in (1.4) arises due to the contribution from animals in which the origin lies in a cycle of length 4, which of course cannot occur in a lattice tree. It is in this way that the strict inequality $z_c^{(a)} < z_c^{(t)}$ [9] (equivalently $\lambda_0 < \lambda_b$) first manifests itself in the $1/d$ expansions.

Much has been proved about lattice trees and lattice animals above the upper critical dimension $d_c = 8$, using the lace expansion. The lace expansion was first adapted to lattice trees and lattice animals in [13]. For sufficiently high dimensions, it has been proved that $t_n \sim A \lambda_0^n n^{-3/2}$ and that the length scale of an n -bond lattice tree is typically of order $n^{1/4}$ [14]. Much stronger results relate the scaling limit of high-dimensional lattice trees to super-Brownian motion [6, 20, 30].

The proof of Theorem 1.1 relies heavily on the lace expansions for lattice trees and lattice animals, and in particular on estimates of [12, 13]. The lace expansions are expansions for the two-point functions

$$G_z^{(t)}(x) = \sum_{n=0}^{\infty} t_n(x) z^n, \quad G_z^{(a)}(x) = \sum_{n=0}^{\infty} a_n(x) z^n, \quad (1.9)$$

where $t_n(x)$ and $a_n(x)$ respectively denote the number of n -bond lattice trees and n -bond lattice

animals containing the two points $0, x \in \mathbb{Z}^d$. Equivalently,

$$G_z(x) = \sum_{C \ni 0, x} z^{|C|}, \quad (1.10)$$

where the sum is over lattice trees or lattice animals containing x, y , according to which model is considered, and where $|C|$ denotes the number of bonds in C .

To prove Theorem 1.1, it is not enough just to have an expansion for the two-point function: an expansion for the one-point function is needed as well. This is a difficulty for lattice trees and lattice animals that does not occur for self-avoiding walk or percolation. In this paper, we develop a new expansion for the one-point function, based on inclusion-exclusion.

The lace expansion and the expansion we present here for the one-point function have been developed so far only in the context of bond trees and bond animals. To apply our approach to related models, such as site animals or site trees, it would be necessary to extend the expansions to these models, and also to extend the estimates of Section 5 below to these models.

Theorem 1.1 first appeared in the PhD thesis [25]; the proof here has been reorganised and simplified.

2 Recursive structure of the proof

The *susceptibility* χ is defined, for lattice trees or lattice animals, by

$$\chi(z) = \sum_{x \in \mathbb{Z}^d} G_z(x). \quad (2.1)$$

For $z \in [0, z_c]$, the lace expansion of [13] expresses χ in terms of another function $\hat{\Pi}_z$ (discussed below in Section 4) via

$$\chi(z) = \frac{g(z) + \hat{\Pi}_z}{1 - 2dz(g(z) + \hat{\Pi}_z)}. \quad (2.2)$$

For d sufficiently large, the susceptibility has been proven to diverge at z_c [12, 13], and this is reflected by the vanishing of the denominator of the right-hand side of (2.2) when $z = z_c$ (see [12, (1.30)]), namely

$$1 - 2dz_c(g_c + \hat{\Pi}_{z_c}) = 0. \quad (2.3)$$

We rewrite (2.3) as

$$z_c = \frac{1}{2d} \frac{1}{g_c + \hat{\Pi}_{z_c}}, \quad (2.4)$$

which expresses z_c in terms of g_c and $\hat{\Pi}_{z_c}$.

Our main tool in obtaining rigorous error estimates is given in Lemma 5.1 below. Using this lemma, we prove the following expansions for $G_{z_c}(s)$ and for $\hat{\Pi}_{z_c}$, where $s \in \mathbb{Z}^d$ is a neighbour of the origin. Recall that $\mathbb{1}_a$ equals 1 for lattice animals and equals 0 for lattice trees.

Theorem 2.1. *Let $s \in \mathbb{Z}^d$ be a neighbour of the origin. For lattice trees or lattice animals,*

$$G_{z_c}(s) = e \left[\frac{1}{2d} + \frac{\frac{7}{2}}{(2d)^2} \right] + o(2d)^{-2}. \quad (2.5)$$

Theorem 2.2. *For lattice trees or lattice animals,*

$$\hat{\Pi}_{z_c} = e \left[-\frac{3}{2d} - \frac{\frac{27}{2} - \mathbb{1}_a \frac{3}{2} e^{-1}}{(2d)^2} \right] + o(2d)^{-2}. \quad (2.6)$$

Our method of proof follows a recursive procedure in which the calculation of the terms in the expansion for z_c is intertwined with the computation of the terms in the expansions for $G_{z_c}(s)$, $\hat{\Pi}_{z_c}$ and g_c . With (1.6) as our starting point, we compute the first terms on the right-hand sides of (2.5)–(2.6), then use the result of that computation together with our new expansion for the one-point function to compute the second term of (1.5), and then from (2.4) obtain the second term of (1.4). We then repeat the process, obtaining an additional term for $\hat{\Pi}_{z_c}$, then an additional term for g_c . Once we have proved Theorem 2.2 and (1.5), the expansion (1.4) follows immediately by substitution into (2.4). The procedure is depicted in Figure 1. Due to its algorithmic nature there is no reason in principle why further terms could not be computed with further effort.

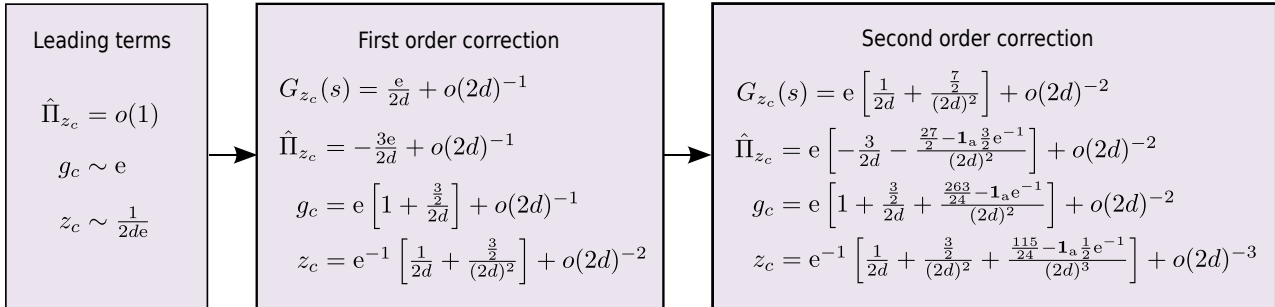


Figure 1: Flow of the proof of Theorem 1.1.

3 Expansion for one-point function

In this section, we develop a new expansion for the one-point functions of lattice trees and lattice animals, simultaneously. The expansion may be considered as a systematic use of inclusion-exclusion to compare with the mean-field model of lattice trees of [3], which is based on the Galton–Watson branching process with critical Poisson offspring distribution.

3.1 Estimate for the one-point function

We begin by stating the one result from Section 3, in Theorem 3.1 below, that will be used later in the proof of Theorem 1.1. The proof of Theorem 3.1 uses *only* the leading behaviour (1.6) of z_c and g_c , together with the important Lemma 5.1 which is used to bound errors.

In the case of $g^{(a)}(z)$, it is convenient to separate the sum over lattice animals depending on whether the origin is contained in a cycle or not, which we denote by $0 \in \text{cycle}$ and $0 \notin \text{cycle}$, respectively. For the former, we define

$$g_o(z) = \mathbb{1}_a \sum_{A \ni 0: 0 \in \text{cycle}} z^{|A|}. \quad (3.1)$$

Then we obtain, for either model,

$$g(z) = \sum_{C \ni 0} z^{|C|} = \sum_{C \ni 0: 0 \notin \text{cycle}} z^{|C|} + g_o(z), \quad (3.2)$$

where the clusters C are lattice trees or lattice animals depending on which model we consider. We will expand the first term on the right-hand side of (3.2), but do not expand $g_o(z)$.

We introduce the notion of a *planted* tree or animal as one which contains the origin as a vertex of degree 1. An important role will be played by the generating function

$$r(z) = \sum_{S \ni s} z^{|S|}, \quad r_c = r(z_c), \quad (3.3)$$

for clusters planted via the bond $\{0, s\}$ with s a specific neighbour of the origin (by symmetry $r(z)$ does not depend on the choice of s). We emphasise that in (3.3) we are abusing notation by writing $S \ni s$ to denote that the *bond* $\{0, s\}$ is contained in S ; we will continue to use this notational convention. Using inclusion-exclusion to account for the fact that removal of the bond $\{0, s\}$ (but not its vertices) from a planted cluster $S \ni s$ must disconnect 0 from the rest of S , we obtain the identity

$$r(z) = zg(z) - zG_z(s). \quad (3.4)$$

At the critical value z_c , we can use (2.3) to replace g by $\hat{\Pi}$ on the right-hand side, and obtain

$$2dr_c = 1 - 2dz_c \hat{\Pi}_{z_c} - 2dz_c G_{z_c}(s). \quad (3.5)$$

The identity (3.5) will be useful in conjunction with the following theorem.

Theorem 3.1. *For lattice trees or lattice animals,*

$$g_c = e^{2dr_c} \left[1 - \frac{1}{2}(2d)r_c^2 + \frac{1}{8}(2d)^2 r_c^4 - \frac{\frac{7}{6}}{(2d)^2} \right] + g_o(z_c) + o(2d)^{-2}. \quad (3.6)$$

The proof of Theorem 3.1 will be discussed at the end of Section 3. It is based on the expansion for g which we discuss next. The remainder of Section 3 is needed only for the proof of Theorem 3.1.

3.2 Expansion for the one-point function

The one-point function for trees, and for animals in which the origin does not belong to a cycle, have the following similar structure. A tree T , or an animal A for which the origin is not in a cycle, consists either of the single vertex 0, or of some number $m \in \{1, \dots, 2d\}$ of planted clusters S_i which intersect pairwise only at the origin. This is depicted in Figure 2.

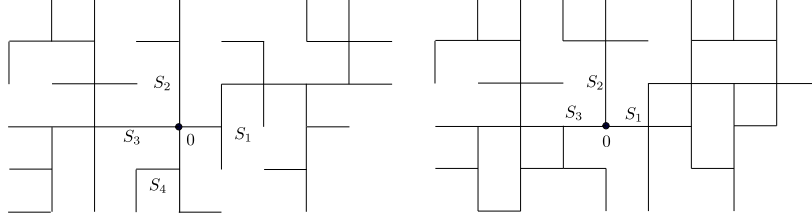


Figure 2: Decomposition into planted clusters S_i , for a lattice tree and for a lattice animal with $0 \notin \text{cycle}$.

Given $0 \leq i < j \leq m$ and a set $\vec{S} = \{S_1, \dots, S_m\}$ of m planted clusters, we define

$$\mathcal{V}_{ij}(\vec{S}) = \begin{cases} -1 & \text{if } S_i \text{ and } S_j \text{ share a common vertex other than } 0 \\ 0 & \text{if } S_i \text{ and } S_j \text{ share no common vertex other than } 0. \end{cases} \quad (3.7)$$

Let $\mathcal{E} = \{e_1, e_2, \dots, e_{2d}\}$ consist of the $2d$ nearest neighbours of the origin ordered such that $e_i = (0, \dots, 1, 0, \dots, 0)$, where the 1 is located at the i -th coordinate for $1 \leq i \leq d$, and $e_i = -e_{i-d}$ for $d+1 \leq i \leq 2d$. Then we can rewrite the one-point function as

$$g(z) = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \sum_{S_1 \ni s_1} z^{|S_1|} \dots \sum_{S_m \ni s_m} z^{|S_m|} \prod_{1 \leq i < j \leq m} (1 + \mathcal{V}_{ij}) + g_o(z). \quad (3.8)$$

The factor $(1 + \mathcal{V}_{ij})$ ensures that S_i and S_j do not intersect each other except at the origin; in particular, this excludes the possibility that $s_i = s_j$. It also ensures that the sum over m in (3.8) is actually a finite sum, since the terms vanish for $m > 2d$.

It follows easily by induction on $n \geq 0$ that

$$\prod_{i=1}^n (1 + x_i) = 1 + \sum_{i=1}^n x_i \prod_{j=i+1}^n (1 + x_j). \quad (3.9)$$

Here, as is the case throughout the paper, the empty product is equal to 1. We iterate (3.9) by applying it to the product that appears on its right-hand side, and obtain

$$\prod_{i=1}^n (1 + x_i) = 1 + \sum_{i=1}^n x_i + \sum_{i=1}^n \sum_{j=i+1}^n x_i x_j \prod_{k=j+1}^n (1 + x_k). \quad (3.10)$$

We expand the product $\prod_{0 \leq i < j \leq m} (1 + \mathcal{V}_{ij})$ in (3.8) using (3.10), as follows. For $m \geq 2$, we define

$$A_{ij} = A_{ij}(m) = \{(i, l) : j < l \leq m\} \cup \{(k, l) : i < k < l \leq m\}, \quad (3.11)$$

and apply (3.10) to rewrite the product $\prod_{0 \leq i < j \leq m} (1 + \mathcal{V}_{ij})$ as

$$\prod_{1 \leq i < j \leq m} (1 + \mathcal{V}_{ij}) = 1 + \sum_{1 \leq i < j \leq m} \mathcal{V}_{ij} + \sum_{1 \leq i < j \leq m} \sum_{(k, l) \in A_{i, j}} \mathcal{V}_{ij} \mathcal{V}_{kl} \prod_{(p, q) \in A_{k, l}} (1 + \mathcal{V}_{pq}). \quad (3.12)$$

We substitute (3.12) into (3.8) to obtain

$$\begin{aligned}
g(z) &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \sum_{S_1 \ni s_1} z^{|S_1|} \dots \sum_{S_m \ni s_m} z^{|S_m|} \\
&\quad \times \left(1 - \sum_{1 \leq i < j \leq m} (-\mathcal{V}_{ij}) + \sum_{1 \leq i < j \leq m} \sum_{(k,l) \in A_{ij}} \mathcal{V}_{ij} \mathcal{V}_{kl} \prod_{(p,q) \in A_{kl}} (1 + \mathcal{V}_{pq}) \right) + g_o(z) \\
&= \Gamma^{(0)}(z) - \Gamma^{(1)}(z) + \tilde{\Gamma}^{(2)}(z) + g_o(z),
\end{aligned} \tag{3.13}$$

where the first three terms on the right-hand side of this expansion for the one-point function arise from the three terms within parentheses in the second line.

The term $\Gamma^{(0)}(z)$ can be immediately computed. Indeed, by definition, and by (3.3)–(3.4),

$$\Gamma^{(0)}(z) = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \sum_{S_1 \ni s_1} z^{|S_1|} \dots \sum_{S_m \ni s_m} z^{|S_m|} = \sum_{m=0}^{\infty} \frac{1}{m!} (2dr(z))^m = e^{2dr(z)}. \tag{3.14}$$

We further expand $\tilde{\Gamma}^{(2)}(z)$, which by definition is

$$\tilde{\Gamma}^{(2)}(z) = \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \sum_{S_1 \ni s_1} z^{|S_1|} \dots \sum_{S_m \ni s_m} z^{|S_m|} \sum_{1 \leq i < j \leq m} \sum_{(k,l) \in A_{ij}} \mathcal{V}_{ij} \mathcal{V}_{kl} \prod_{(p,q) \in A_{kl}} (1 + \mathcal{V}_{pq}). \tag{3.15}$$

A second application of (3.10), as in (3.12), gives

$$\prod_{(p,q) \in A_{kl}} (1 + \mathcal{V}_{pq}) = 1 + \sum_{(p,q) \in A_{kl}} \mathcal{V}_{pq} + \sum_{(p,q) \in A_{kl}} \sum_{(r,s) \in A_{pq}} \mathcal{V}_{pq} \mathcal{V}_{rs} \mathcal{I}_{rs}, \tag{3.16}$$

where we define

$$\mathcal{I}_{rs} = \prod_{(t,u) \in A_{rs}} (1 + \mathcal{V}_{tu}). \tag{3.17}$$

We substitute (3.16) into (3.15). With (3.13), this leads to the expansion

$$g(z) = \Gamma^{(0)}(z) - \Gamma^{(1)}(z) + \Gamma^{(2)}(z) - \Gamma^{(3)}(z) + \tilde{\Gamma}^{(4)}(z) + g_o(z), \tag{3.18}$$

where

$$\Gamma^{(1)}(z) = \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \sum_{S_1 \ni s_1} z^{|S_1|} \dots \sum_{S_m \ni s_m} z^{|S_m|} \sum_{1 \leq i < j \leq m} (-\mathcal{V}_{ij}), \tag{3.19}$$

$$\Gamma^{(2)}(z) = \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \sum_{S_1 \ni s_1} z^{|S_1|} \dots \sum_{S_m \ni s_m} z^{|S_m|} \sum_{1 \leq i < j \leq m} \sum_{(k,l) \in A_{ij}} \mathcal{V}_{ij} \mathcal{V}_{kl}, \tag{3.20}$$

$$\Gamma^{(3)}(z) = \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \sum_{S_1 \ni s_1} z^{|S_1|} \dots \sum_{S_m \ni s_m} z^{|S_m|} \sum_{1 \leq i < j \leq m} \sum_{(k,l) \in A_{ij}} \sum_{(p,q) \in A_{kl}} (-\mathcal{V}_{ij} \mathcal{V}_{kl} \mathcal{V}_{pq}) \tag{3.21}$$

$$\begin{aligned}
\tilde{\Gamma}^{(4)}(z) &= \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \sum_{S_1 \ni s_1} z^{|S_1|} \dots \sum_{S_m \ni s_m} z^{|S_m|} \sum_{1 \leq i < j \leq m} \sum_{(k,l) \in A_{ij}} \sum_{(p,q) \in A_{kl}} \sum_{(r,s) \in A_{pq}} \mathcal{V}_{ij} \mathcal{V}_{kl} \mathcal{V}_{pq} \mathcal{V}_{rs} \mathcal{I}_{rs}. \\
&\tag{3.22}
\end{aligned}$$

Note that i in $\Gamma^{(i)}$ counts the number of factors of \mathcal{V} in each term, and the remainder term $\tilde{\Gamma}^{(4)}$ also contains the factor \mathcal{I}_{rs} . This last factor could be expanded further, and the process continued indefinitely, but for the proof of Theorem 1.1 the expansion (3.18) suffices.

3.3 Identities and estimates for the one-point function

To analyse the terms $\Gamma^{(2)}(z)$ and $\Gamma^{(3)}(z)$, it is useful to decompose according to the cardinality of the label sets $\{i, j, k, l\}$ and $\{i, j, k, l, p, q\}$ (respectively) and we write $\Gamma^{(m,n)}$ for the contribution to $\Gamma^{(m)}$ arising from label sets of cardinality n . Thus, for $m = 2, 3$, $\Gamma^{(m)} = \sum_n \Gamma^{(m,n)}$, where m counts the number of \mathcal{V} factors and n counts the cardinality of the label set. In particular, when $m = 2$ we have the two possibilities $n = 3, 4$, while for $m = 3$ the possibilities are $n = 3, 4, 5, 6$. As we discuss in more detail below, $\Gamma^{(3,n)}$ is an error term for $n = 4, 5, 6$, as is $\tilde{\Gamma}^{(4)}$. For Theorem 1.1, we will need an accurate calculation of $\Gamma^{(2,3)}(z)$, $\Gamma^{(2,4)}(z)$ and $\Gamma^{(3,3)}(z)$. To obtain convenient expressions for these important terms, we make the definitions

$$Z^{(1)}(z) = \sum_{s_1, s_2 \in \mathcal{E}} \sum_{S_1 \ni s_1} \sum_{S_2 \ni s_2} z^{|S_1|+|S_2|} (-\mathcal{V}_{12}), \quad (3.23)$$

$$Z^{(2)}(z) = \sum_{s_1, s_2, s_3 \in \mathcal{E}} \sum_{S_1 \ni s_1} \sum_{S_2 \ni s_2} \sum_{S_3 \ni s_3} z^{|S_1|+|S_2|+|S_3|} \mathcal{V}_{12} \mathcal{V}_{13}, \quad (3.24)$$

$$Z^{(3)}(z) = \sum_{s_1, s_2, s_3 \in \mathcal{E}} \sum_{S_1 \ni s_1} \sum_{S_2 \ni s_2} \sum_{S_3 \ni s_3} z^{|S_1|+|S_2|+|S_3|} (-\mathcal{V}_{12} \mathcal{V}_{13} \mathcal{V}_{23}). \quad (3.25)$$

Lemma 3.2. *The following identities hold:*

$$\Gamma^{(1)}(z) = \frac{1}{2!} \Gamma^{(0)}(z) Z^{(1)}(z), \quad (3.26)$$

$$\Gamma^{(2,3)}(z) = \frac{3}{3!} \Gamma^{(0)}(z) Z^{(2)}(z), \quad (3.27)$$

$$\Gamma^{(2,4)}(z) = \frac{3}{4!} \Gamma^{(0)}(z) Z^{(1)}(z)^2, \quad (3.28)$$

$$\Gamma^{(3,3)}(z) = \frac{1}{3!} \Gamma^{(0)}(z) Z^{(3)}(z). \quad (3.29)$$

Proof. For $\Gamma^{(1)}(z)$, we interchange the sums over $s_1, \dots, s_m \in \mathcal{E}$ and $1 \leq i < j \leq m$ in (3.19) to obtain

$$\begin{aligned} \Gamma^{(1)}(z) &= \sum_{m=2}^{\infty} \frac{1}{m!} (2dr(z))^{m-2} \sum_{1 \leq i < j \leq m} \sum_{s_i, s_j \in \mathcal{E}} \sum_{S_i \ni s_i} z^{|S_i|} \sum_{S_j \ni s_j} z^{|S_j|} (-\mathcal{V}_{ij}) \\ &= \sum_{m=2}^{\infty} \frac{1}{m!} (2dr(z))^{m-2} \binom{m}{2} Z^{(1)}(z) \\ &= \frac{1}{2} \Gamma^{(0)}(z) Z^{(1)}(z), \end{aligned} \quad (3.30)$$

where we used (3.14) in the last step.

For $\Gamma^{(2,3)}$, the condition $|\{i, j, k, l\}| = 3$ is satisfied when $k = i$, $k = j$ or $l = j$. In all cases, we choose three labels from a set of m and order them; this order automatically determines which one corresponds to i , j , k and l . Hence, the number of options for the labels is $3\binom{m}{3}$. Using symmetry, we obtain

$$\begin{aligned}\Gamma^{(2,3)}(z) &= \sum_{m=3}^{\infty} \frac{1}{m!} (2dr(z))^{m-3} 3\binom{m}{3} \sum_{s_1, s_2, s_3 \in \mathcal{E}} \sum_{S_1 \ni s_1} \sum_{S_2 \ni s_2} \sum_{S_3 \ni s_3} z^{|S_1|+|S_2|+|S_3|} \mathcal{V}_{12} \mathcal{V}_{13} \\ &= \frac{3}{3!} \Gamma^{(0)}(z) Z^{(2)}(z).\end{aligned}\tag{3.31}$$

For the case $\Gamma^{(2,4)}$, the labels i, j, k, l are distinct. To determine the number of possibilities for the labels we chose four labels from a set of m and order them. Then i is the smallest by definition, j has the remaining 3 options, and once j is determined, so are k and l . Hence, there are $3\binom{m}{4}$ possibilities. By interchanging sums and using symmetry, we obtain

$$\begin{aligned}\Gamma^{(2,4)}(z) &= \sum_{m=4}^{\infty} \frac{1}{m!} (2dr(z))^{m-4} 3\binom{m}{4} \sum_{s_1, s_2, s_3, s_4 \in \mathcal{E}} \sum_{S_1 \ni s_1} \sum_{S_2 \ni s_2} \sum_{S_3 \ni s_3} \sum_{S_4 \ni s_4} z^{|S_1|+|S_2|+|S_3|+|S_4|} \mathcal{V}_{12} \mathcal{V}_{34} \\ &= \frac{3}{4!} \Gamma^{(0)}(z) Z^{(1)}(z)^2.\end{aligned}\tag{3.32}$$

For $\Gamma^{(3,3)}$, it must be the case that $i < j < l$, $k = i$, $p = j$ and $q = l$. Thus the number of possibilities for the labels is given by choosing three labels from a set of m and ordering them in this way. By interchanging sums and using symmetry, we obtain

$$\begin{aligned}\Gamma^{(3,3)}(z) &= \sum_{m=3}^{\infty} \frac{1}{m!} (2dr(z))^{m-3} \binom{m}{3} \sum_{s_1, s_2, s_3 \in \mathcal{E}} \sum_{S_1 \ni s_1} \sum_{S_2 \ni s_2} \sum_{S_3 \ni s_3} z^{|S_1|+|S_2|+|S_3|} (-\mathcal{V}_{12} \mathcal{V}_{13} \mathcal{V}_{23}) \\ &= \frac{1}{3!} \Gamma^{(0)}(z) Z^{(3)}(z).\end{aligned}\tag{3.33}$$

This completes the proof. \square

Now we can prove Theorem 3.1, using estimates from Section 8.1. The latter are proved using only the leading behaviour (1.6) of z_c and g_c , together with Lemma 5.1 which gives error estimates.

Proof of Theorem 3.1. It is shown in Lemma 8.2 that the terms $\Gamma^{(3,n)}(z_c)$ ($n = 4, 5, 6$) and $\tilde{\Gamma}^{(4)}(z_c)$ are $O(2d)^{-3}$. In addition, Lemma 8.1 gives expressions for the $Z_c^{(i)} = Z^{(i)}(z_c)$ factors appearing in Lemma 3.2, accurate to within error $o(2d)^{-2}$. We substitute the identities of Lemma 3.2 into (3.18), and apply the results of Lemmas 8.1–8.2 (together with $r_c \leq z_c g_c = O(2d)^{-1}$ by (1.6)), to obtain

$$\begin{aligned}g_c &= e^{2dr_c} \left[1 - \frac{1}{2!} Z_c^{(1)} + \left(\frac{3}{3!} Z_c^{(2)} + \frac{3}{4!} (Z_c^{(1)})^2 \right) - \frac{1}{3!} Z_c^{(3)} \right] + g_o(z_c) + o(2d)^{-2} \\ &= e^{2dr_c} \left[1 - \frac{1}{2} (2d) r_c^2 + \frac{1}{8} (2d)^2 r_c^4 - \frac{7}{6} \frac{1}{(2d)^2} \right] + g_o(z_c) + o(2d)^{-2},\end{aligned}\tag{3.34}$$

and the proof is complete. \square

4 Lace expansion

We recall some fundamental facts about the lace expansion for lattice trees and lattice animals from [13] (see also [12, 30]).

4.1 Lace expansion for lattice trees

A lattice tree containing $0, x$, which contributes to the two-point function $G_z(x) = \sum_{T \ni 0, x} z^{|T|}$ of (1.10), can be decomposed into a unique path joining 0 and x , which we call the *backbone*, together with the disjoint collection of subtrees consisting of the connected components that remain after the bonds in the backbone (but not the vertices) are removed. We refer to the subtrees (which may consist of a single vertex) as *ribs*. The definitions should be clear from Figure 3.

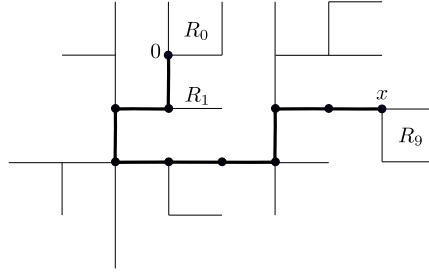


Figure 3: Decomposition of a lattice tree T into the backbone from 0 to x (bold) and the ribs $\vec{R} = \{R_0, \dots, R_9\}$.

By definition, the ribs are mutually avoiding. However, in high dimensions, if this avoidance restriction were relaxed then intersections between ribs should be in some sense still be rare. The lace expansion is a way of making this vague intuition precise, via a systematic use of inclusion exclusion. To describe the basic idea, we need the following definitions.

Let $D : \mathbb{Z}^d \rightarrow \mathbb{R}$ denote the one-step transition probability function for simple random walk on \mathbb{Z}^d , i.e.,

$$D(x) = \begin{cases} (2d)^{-1} & \text{if } \|x\|_1 = 1 \\ 0 & \text{otherwise.} \end{cases} \quad (4.1)$$

The *convolution* of absolutely summable functions $f : \mathbb{Z}^d \rightarrow \mathbb{R}$ and $h : \mathbb{Z}^d \rightarrow \mathbb{R}$ is given by

$$(f * h)(x) = \sum_{y \in \mathbb{Z}^d} f(y)h(x - y). \quad (4.2)$$

If it were the case that the rib R_0 were permitted to intersect the remaining ribs, then the two-point function $G_z^{(t)}(x)$ (for $x \neq 0$) would be given by the convolution

$$g^{(t)}(z)(2dzD) * G_z^{(t)}(x) = g^{(t)}(z) \sum_{y \in \mathbb{Z}^d} 2dzD(y)G_z^{(t)}(x - y), \quad (4.3)$$

where the factor $g^{(t)}(z)$ captures the rib at the origin, y is the location of the next vertex after 0 along the backbone, and $G_z^{(t)}(x - y)$ captures the backbone from y to x together with its ribs. Compared to the two-point function, (4.3) permits disallowed intersections and thus includes too much. In fact, it provides the basis of the mean-field model introduced in [5] and further studied in [3, 30]. The lace expansion corrects the overcounting in (4.3) with the help of the function $\Pi_z : \mathbb{Z}^d \rightarrow \mathbb{R}$ which appears in the *identity*

$$G_z^{(t)}(x) = \delta_{0,x} g^{(t)}(z) + \Pi_z^{(t)}(x) + g^{(t)}(z) 2dzD * G_z^{(t)}(x) + \Pi_z^{(t)} * 2dzD * G_z^{(t)}(x). \quad (4.4)$$

In [13], an expansion for $\hat{\Pi}_z^{(t)} = \sum_{x \in \mathbb{Z}^d} \Pi_z^{(t)}(x)$ is given, of the form

$$\hat{\Pi}_z^{(t)} = \sum_{N=1}^{\infty} (-1)^N \hat{\Pi}_z^{(t,N)}. \quad (4.5)$$

It is known (see [12]) that there is a $c > 0$ such that for all $N \geq 1$ and all $z \in [0, z_c]$,

$$0 \leq \hat{\Pi}_z^{(t,N)} \leq c^N d^{-N} \quad (4.6)$$

and this implies that the only terms that can contribute to (2.6) for lattice trees are those with $N = 1, 2$. We define these terms next.

We define $\mathcal{U}_{ij}(\vec{R})$ by

$$\mathcal{U}_{ij}(\vec{R}) = \begin{cases} -1 & \text{if ribs } R_i \text{ and } R_j \text{ share a common vertex} \\ 0 & \text{if ribs } R_i \text{ and } R_j \text{ share no common vertex.} \end{cases} \quad (4.7)$$

Let $\mathcal{W}(x)$ denote the set of simple random walk paths ω from 0 to x , i.e., sequences $x_0 = 0, x_1, \dots, x_n = x$ with $\|x_{i+1} - x_i\|_1 = 1$ for all i , for any $n = |\omega| \geq 0$. The function $\Pi_z^{(t,1)}(x)$ is defined by

$$\Pi_z^{(t,1)}(x) = \sum_{\omega \in \mathcal{W}(x): |\omega| \geq 1} z^{|\omega|} \sum_{R_0 \ni \omega(0)} z^{|R_0|} \dots \sum_{R_{|\omega|} \ni x} z^{|R_{|\omega|}|} (-\mathcal{U}_{0|\omega|}) \prod_{\substack{0 \leq i < j \leq |\omega| \\ (i,j) \neq (0,|\omega|)}} (1 + \mathcal{U}_{ij}). \quad (4.8)$$

For a nonzero contribution, the factor $\mathcal{U}_{0|\omega|}$ forces the first and last ribs to intersect, while the final product disallows all other intersection among the ribs. The function $\Pi_z^{(t,2)}(x)$ is defined by

$$\Pi_z^{(t,2)}(x) = \sum_{\omega \in \mathcal{W}(x): |\omega| \geq 2} z^{|\omega|} \sum_{R_0 \ni \omega(0)} z^{|R_0|} \dots \sum_{R_{|\omega|} \ni x} z^{|R_{|\omega|}|} \sum_{L \in \mathcal{L}^{(2)}[0, |\omega|]} \prod_{ij \in L} \mathcal{U}_{ij} \prod_{i'j' \in \mathcal{C}(L)} (1 + \mathcal{U}_{i'j'}), \quad (4.9)$$

where the set $\mathcal{L}^{(2)}[0, |\omega|]$ of (2-edge) *laces* is given by:

$$\mathcal{L}^{(2)}[0, n] = \{ \{0j, jn\} : 0 < j < n \} \cup \{ \{0j, in\} : 0 < i < j < n \}, \quad (4.10)$$

and where the set $\mathcal{C}(L)$ *compatible* with $L \in \mathcal{L}^{(2)}[0, n]$ is given:

- (i) for $L = \{0j, jn\}$, by all pairs kl with $0 \leq k < l \leq n$ except $0l$ with $l > j$ and kn with $k < j$;
- (ii) for $L = \{0j, in\}$ with $i < j$, by all pairs kl excepting both $0l$ with $l > j$ and kn with $k < i$.

For more details, see [13] or [12, 30].

4.2 Lace expansion for lattice animals

The two-point function $G_z^{(a)}(x) = \sum_{A \ni 0, x} z^{|A|}$ for lattice animals was defined in (1.10) as the sum over lattice animals that contain both vertices 0 and x . An animal A with this characteristic contains a path connecting 0 to x ; however, unlike the lattice tree case, this path is not necessarily unique. To deal with this we use the following definitions.

Let A be an animal containing the vertices x and y . We say that A has a *double connection* from x to y if there are two bond-disjoint self avoiding walks in A between x and y (the walks may share a common vertex, but not a common bond), or if $x = y$. The set of all animals having a double connection between x and y is denoted by $\mathcal{D}_{x,y}$. A bond $\{x, y\}$ in A is *pivotal* for the connection from x to y , if its removal would disconnect the animal into two connected components with x contained in one of them and y in the other.

An animal $A \ni x, y$ that is not an element of $\mathcal{D}_{x,y}$ has at least one pivotal bond for the connection from x to y . To establish an order among these edges, we define the *first* pivotal bond to be the unique bond for which there is a double connection between x and one of the endpoints of this bond. This endpoint is the *first* endpoint of the first pivotal bond. To determine the *second* pivotal bond, the role of x is played by the second endpoint of the first pivotal bond, and so on.

For a lattice animal A that contains x and y , the *backbone* is the ordered set of oriented pivotal bonds for the connection from x to y . The backbone is not necessarily connected. The *ribs* are the connected components that remain after the bonds in the backbone (but not the vertices) are removed from A . By definition, the ribs are doubly connected between the corresponding backbone vertices, and are mutually avoiding. See Figure 4 for an example.

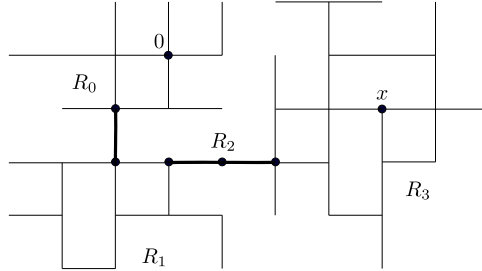


Figure 4: Decomposition of a lattice animal A into the backbone from 0 to x (bold), and the ribs $\vec{R} = \{R_0, R_1, R_2, R_3\}$. The rib R_2 consists only of the vertex in the backbone.

Let B be an arbitrary finite ordered set of directed bonds

$$B = ((u_1, v_1), \dots, (u_{|B|}, v_{|B|})) ,$$

and let $v_0 = 0$ and $u_{|B|+1} = x$. Then we can regard the two-point function as a sum over the backbone B and mutually nonintersecting ribs $\vec{R} = \{R_0, \dots, R_{|B|}\}$. It is shown in [13] how to apply inclusion exclusion to obtain an identity

$$G_z^{(a)}(x) = \delta_{0,x} g^{(a)}(z) + \Pi_z^{(a)}(x) + g^{(a)}(z) * 2dzD * G_z^{(a)}(x) + \Pi_z^{(a)} * 2dzD * G_z^{(a)}(x), \quad (4.11)$$

with $\Pi_z^{(a)}$ given by the alternating series

$$\hat{\Pi}_z^{(a)} = \sum_{N=0}^{\infty} (-1)^N \hat{\Pi}_z^{(a,N)}. \quad (4.12)$$

It is known (see [12]) that there is a $c > 0$ such that for all $N \geq 0$ and all $z \in [0, z_c]$,

$$0 \leq \hat{\Pi}_z^{(a,N)} \leq c^N d^{-(N \vee 1)} \quad (4.13)$$

and this implies that the only terms that can contribute to (2.6) for lattice animals are those with $N = 0, 1, 2$.

The following explicit formulas are obtained in [13]. First,

$$\Pi_z^{(a,0)}(x) = (1 - \delta_{0,x}) \sum_{R \in \mathcal{D}_{0,x}} z^{|R|}. \quad (4.14)$$

With $\mathcal{U}_{ij}(\vec{R})$ as in (4.7) but for the new notion of ribs \vec{R} ,

$$\Pi_z^{(a,1)}(x) = \sum_{B:|B| \geq 1} z^{|B|} \left[\prod_{k=0}^{|B|} \sum_{R_k \in \mathcal{D}_{v_k, u_{k+1}}} z^{|R_k|} \right] (-\mathcal{U}_{0,|B|}) \prod_{\substack{0 \leq i < j \leq |B| \\ (i,j) \neq (0,|B|)}} (1 + \mathcal{U}_{st}), \quad (4.15)$$

with $v_0 = 0$ and $u_{|B|+1} = x$. The factor $\mathcal{U}_{0,|B|}$ in the previous expression forces an intersection between the first and last ribs, and the last product forbids all other rib intersections. Finally,

$$\Pi_z^{(a,2)}(x) = \sum_{B:|B| \geq 1} z^{|B|} \left[\prod_{k=0}^{|B|} \sum_{R_k \in \mathcal{D}_{v_k, u_{k+1}}} z^{|R_k|} \right] \sum_{L \in \mathcal{L}^{(2)}[0,|B|]} \prod_{ij \in L} \mathcal{U}_{ij} \prod_{i'j' \in \mathcal{C}(L)} (1 + \mathcal{U}_{i'j'}), \quad (4.16)$$

with $\mathcal{L}^{(2)}$ and $\mathcal{C}(L)$ as defined around (4.10).

5 Fourier estimates

In this section, we formulate an essential ingredient for the error estimates in Theorem 1.1, in Lemma 5.1 below. The proof is based on the Fourier transform.

The *Fourier transform* of an absolutely summable function $f : \mathbb{Z}^d \rightarrow \mathbb{C}$ is defined by

$$\hat{f}(k) = \sum_{x \in \mathbb{Z}^d} f(x) e^{ik \cdot x}, \quad (5.1)$$

where $k \in [-\pi, \pi]^d$ and $k \cdot x = \sum_{j=1}^d k_j x_j$. For example, the transition probability D of (4.1) has Fourier transform $\hat{D}(k) = d^{-1} \sum_{j=1}^d \cos k_j$. The *inverse Fourier transform*, which recovers f from \hat{f} , is given by

$$f(x) = \int_{[-\pi, \pi]^d} \hat{f}(k) e^{-ik \cdot x} \frac{dk}{(2\pi)^d}. \quad (5.2)$$

Recall that the convolution of the functions f and g was defined in (4.2). We denote by f^{*l} the convolution of l factors of f , i.e.,

$$f^{*l}(x) = \underbrace{(f * f * \cdots * f)}_l(x).$$

The Fourier transform of a convolution is the product of Fourier transforms: $\widehat{f * g} = \hat{f}\hat{g}$.

In this notation, $D^{*l}(x)$ is the l -step transition probability that simple random walk travels from 0 to x in l steps. We take $f = D^{*2m}$ and $x = 0$ in (5.2) to obtain

$$D^{*2m}(0) = \int_{[-\pi, \pi]^d} \hat{D}(k)^{2m} \frac{dk}{(2\pi)^d}. \quad (5.3)$$

A proof of the elementary fact that $D^{*2m}(0) \leq C_m(2d)^{-m}$ for some constant C_m (uniformly in d) can be found in [19, (3.12)]. Therefore,

$$\int_{[-\pi, \pi]^d} \hat{D}(k)^{2m} \frac{dk}{(2\pi)^d} \leq \frac{C_m}{(2d)^m}. \quad (5.4)$$

The *infrared bound* for nearest-neighbour lattice trees and lattice animals, given in [12, (1.25)], states that for dimensions $d \geq d_0$ (for some sufficiently large d_0), there is a positive constant c independent of z and d , such that for $0 \leq z \leq z_c$,

$$0 \leq \hat{G}_z(k) \leq \frac{cd}{|k|^2}, \quad (5.5)$$

where $|k| = (k_1^2 + \cdots + k_d^2)^{1/2}$. The definition of $\hat{G}_z(k)$ requires some care when $z = z_c$, because $G_{z_c}(x)$ is not summable. Nevertheless it is possible to define $\hat{G}_{z_c}(k)$ in a natural way such that its inverse Fourier transform is $G_{z_c}(x)$. The subtleties associated with this point are discussed in [12, Appendix A].

Let i be a non-negative integer and let C be a cluster (a tree or an animal) containing the vertices x and y . We denote by

$$\{x \overset{i}{\leftrightarrow} y\} \quad (5.6)$$

the event that there exists a self-avoiding path in C , of length at least i , connecting x and y . We define the two-point function for clusters in which x is connected to y by a path of length at least i by

$$G_z^{(i)}(x) = \sum_{C \ni 0, x : 0 \overset{i}{\leftrightarrow} x} z^{|C|}. \quad (5.7)$$

Then

$$G_z(x) = G_z^{(0)}(x) = g(z)\delta_{0,x} + G_z^{(1)}(x),$$

since for $x = 0$ the two-point function $G_z(x)$ reduces to the one-point function $g(z)$, and for $x \neq 0$ a path connecting these two vertices requires at least one step.

For integers $m, n \geq 1$, and vertices $x, y \in \mathbb{Z}^d$, we define

$$S_z^{(m,n)}(x) = \sum_{i_1 + \dots + i_n = m} G_z^{(i_1)} * \dots * G_z^{(i_n)}(x), \quad (5.8)$$

where the sum is over nonnegative i_1, \dots, i_n . Let

$$S_z^{(m,n)} = \sup_{x \in \mathbb{Z}^d} S_z^{(m,n)}(x). \quad (5.9)$$

The statement and proof of the following lemma are closely related to [19, Lemma 3.1].

Lemma 5.1. *Let m and n be non-negative integers and let $d > \max\{d_0, 4n\}$. There is a constant $C_{m,n}$, whose value depends only on m and n , such that*

$$S_{z_c}^{(m,n)} \leq \frac{C_{m,n}}{(2d)^{m/2}}. \quad (5.10)$$

Proof. We first prove that there is a constant $K_{m,n}$

$$\sup_x (D^{*m} * G_{z_c}^{*n})(x) \leq \frac{K_{m,n}}{(2d)^{m/2}}. \quad (5.11)$$

Using the inverse Fourier transform (5.2) and $\widehat{f * g} = \hat{f} \hat{g}$, we have

$$D^{*m} * G_{z_c}^{*n}(x) = \int_{[-\pi, \pi]^d} \hat{D}(k)^m \hat{G}_{z_c}(k)^n e^{-ik \cdot x} \frac{dk}{(2\pi)^d}.$$

By the Cauchy–Schwarz inequality,

$$D^{*m} * G_{z_c}^{*n}(x) \leq \left(\int_{[-\pi, \pi]^d} \hat{D}(k)^{2m} \frac{dk}{(2\pi)^d} \right)^{1/2} \left(\int_{[-\pi, \pi]^d} \hat{G}_{z_c}(k)^{2n} \frac{dk}{(2\pi)^d} \right)^{1/2}. \quad (5.12)$$

Then (5.4) gives (5.11), once we show that the second factor on the right-hand side of (5.12) is bounded uniformly in large d . By (5.5), it suffices to verify that the integral

$$I_{d,n} = \int_{[-\pi, \pi]^d} \frac{d^{2n}}{|k|^{4n}} \frac{dk}{(2\pi)^d}, \quad (5.13)$$

which is finite for $d > 4n$, is monotone nonincreasing in d .

This monotonicity has been encountered many times previously in the literature (e.g., [19]), and can be proved as follows. For $A > 0$ and $j > 0$, a change of variables in the integral leads to

$$\frac{1}{A^j} = \frac{1}{\Gamma(j)} \int_0^\infty u^{j-1} e^{-uA} du. \quad (5.14)$$

We apply this identity with $A = d^{-1}|k|^2$ and $j = 2n$, and then use Fubini's theorem to obtain

$$\begin{aligned} I_{d,n} &= \frac{1}{\Gamma(2n)} \int_{[-\pi, \pi]^d} \int_0^\infty u^{2n-1} e^{-u|k|^2/d} du \frac{dk}{(2\pi)^d} \\ &= \frac{1}{\Gamma(2n)} \int_0^\infty u^{2n-1} \left(\int_{-\pi}^\pi e^{-ut^2/d} \frac{dt}{2\pi} \right)^d du = \frac{1}{\Gamma(2n)} \int_0^\infty u^{2n-1} \|f_u\|_{1/d} du, \end{aligned} \quad (5.15)$$

where $f_u(t) = e^{-ut^2}$ and $\|f\|_p = (\int_{-\pi}^{\pi} f(t)^p dt/2\pi)^{1/p}$. Since $dt/2\pi$ is a probability measure on $[-\pi, \pi]$,

$$\|f\|_{1/(d+1)} \leq \|f\|_{1/d}. \quad (5.16)$$

Therefore, as required, $I_{d+1,n} \leq I_{d,n}$, and the proof of (5.11) is complete.

Turning now to (5.10), we first consider the case of lattice trees. In (5.7), if we neglect the self-avoidance restriction among the first i steps in the path connecting x and y , and treat the first i ribs as independent of each other and of the subtree that comes after the i^{th} step, we obtain the upper bound

$$G_z^{(i)}(x) \leq (2dzg(z))^i (D^{*i} * G_z)(x). \quad (5.17)$$

For the case of lattice animals, the same bound is plausible and indeed also holds; this can be seen using a small modification in the proof of [13, Lemma 2.1]. With the definition of $S_z^{(m,n)}(x)$ in (5.8), this implies that for either model

$$S_z^{(m,n)}(x) = \sum_{i_1 + \dots + i_n = m} G_z^{(i_1)} * \dots * G_z^{(i_n)}(x) \leq \tilde{C}_{m,n} (2dzg(z))^m D^{*m} * G_z^{*n}(x), \quad (5.18)$$

where $\tilde{C}_{m,n}$ is the number of terms in the sum (its exact value is unimportant). By (1.6), $2dz_c g_c \leq 2$ for d large enough. Together with (5.11), this implies that

$$S_{z_c}^{(m,n)}(x) \leq \tilde{C}_{m,n} 2^m \frac{K_{m,n}}{(2d)^{m/2}},$$

and the proof is complete. \square

6 Second term

In this section, we compute the $(2d)^{-2}$ term in the expansion for z_c in (1.4), and the $(2d)^{-1}$ term in the expansion for g_c in (1.5). We follow the strategy discussed in Section 2: we first compute the $(2d)^{-1}$ terms in the expansions for $G_{z_c}(s)$ and for Π_{z_c} in (2.5)–(2.6), then use this to compute the desired term for g_c , and finally obtain the desired term for z_c .

A useful quantity is

$$Q(x) = \sum_{C_0 \ni 0} \sum_{C_x \ni x} z_c^{|C_0| + |C_x|} \mathbb{1}_{C_0 \cap C_x \neq \emptyset}, \quad (6.1)$$

where the sum is over clusters (both trees or both animals) containing 0 and x , respectively. It is shown in Lemma 8.3 that for x a neighbour s of the origin, and for both lattice trees and lattice animals, the leading behaviour

$$Q(s) = 2z_c^2 g_c^3 + o(2d)^{-1} = \frac{2e}{2d} + o(2d)^{-1} \quad (6.2)$$

arises from the presence of the bond $\{0, s\}$ in one of the two clusters C_0 or C_1 . The proof of Lemma 8.3 uses only (1.6) and Lemma 5.1.

Lemma 6.1. *For lattice trees or lattice animals, and for a neighbour s of the origin,*

$$G_{z_c}(s) = \frac{e}{2d} + o(2d)^{-1}. \quad (6.3)$$

Proof. For a lattice tree or lattice animal containing $0, s$, either the bond $\{0, s\}$ is occupied or it is not. In the latter case, there must be an occupied path of length at least 3. In the former case, we overcount with independent clusters at 0 and s . This gives

$$G_{z_c}(s) \leq z_c g_c^2 + G_{z_c}^{(3)}(s) \leq z_c g_c^2 + S_{z_c}^{(3,1)}, \quad (6.4)$$

where the last inequality comes from (5.8). By (1.6) and Lemma 5.1, it follows that

$$G_{z_c}(s) \leq \frac{e}{2d} + o(2d)^{-1}. \quad (6.5)$$

For a lower bound, we consider only the case where $\{0, s\}$ is occupied and not part of a cycle (for lattice animals). It follows from inclusion exclusion that

$$G_{z_c}(s) \geq z_c g_c^2 - z_c Q(s), \quad (6.6)$$

and it then follows from (6.2) and (1.6) that

$$G_{z_c}(s) \geq \frac{e}{2d} + o(2d)^{-1}. \quad (6.7)$$

This completes the proof. \square

Lemma 6.2. *For lattice trees or lattice animals,*

$$\hat{\Pi}_{z_c} = -\frac{3e}{2d} + o(2d)^{-1}. \quad (6.8)$$

Proof. It follows from (4.6) and (4.13) that we need only consider the contributions due to $\hat{\Pi}_{z_c}^{(t,1)}$ for trees, and due to $\hat{\Pi}_{z_c}^{(a,0)}$ and $\hat{\Pi}_{z_c}^{(a,1)}$ for animals, since larger values of N contribute $O(2d)^{-2}$. Moreover, we can neglect $\hat{\Pi}_{z_c}^{(a,0)}$. To see this, we recall the definition (4.14) and apply the BK inequality of [13, Lemma 2.1] and Lemma 5.1 to see that

$$\hat{\Pi}_{z_c}^{(a,0)} \leq \sum_{i+j=4} \sum_{x \in \mathbb{Z}^d} G_{z_c}^{(i)}(x) G_{z_c}^{(j)}(x) = S_{z_c}^{(4,2)} \leq O(2d)^{-2}, \quad (6.9)$$

where the restriction to $i + j = 4$ arises because only animals in which the origin is in a cycle of length at least 4 can occur. Therefore, we can restrict attention to the case $N = 1$.

By definition,

$$\hat{\Pi}_{z_c}^{(1)} = \Pi_{z_c}^{(1)}(0) + \sum_{s: \|s\|_1=1} \Pi_{z_c}^{(1)}(s) + \sum_{x: \|x\|_1 \geq 2} \Pi_{z_c}^{(1)}(x). \quad (6.10)$$

A nonzero contribution to $\Pi_{z_c}^{(1)}(x)$ requires the existence of three bond-disjoint paths as indicated in Figure 5 (with $y = 0$ or $y = x$ allowed), to ensure that $\mathcal{U}_{0|\omega|} = -1$ in (4.8) or $\mathcal{U}_{0|B|} = -1$ in (4.15). This implies that

$$\hat{\Pi}_{z_c}^{(1)} \leq \sum_{x, y \in \mathbb{Z}^d} G_{z_c}(x) G_{z_c}(y) G_{z_c}(y - x) = S_{z_c}^{(0,3)}(0) \leq S_{z_c}^{(0,3)}; \quad (6.11)$$

a detailed derivation of this estimate can be found, e.g., in [30, Theorem 8.2]. The crude bound (6.11) can be greatly improved by replacing two-point functions by factors $G_{z_c}^{(i)}$ when there must be at least i steps taken. In this way, for contributions to $\hat{\Pi}_{z_c}^{(1)}$ in which there must exist paths from 0 to x , from 0 to y , and from x to y , of total length at least m , we can improve the upper bound $S_{z_c}^{(0,3)}$ to $S_{z_c}^{(m,3)} \leq O(2d)^{-m/2}$. In particular, this implies that the last sum on the right-hand side of (6.10) is bounded by $S_{z_c}^{(4,3)} \leq O(2d)^{-2}$ and thus is an error term.

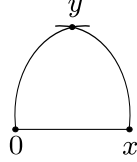


Figure 5: Intersection required for a nonzero contribution to $\Pi_z^{(1)}(x)$.

The leading behaviour arises from the other two terms. We consider both trees and animals simultaneously. Consider first the lower bound. For $\Pi_{z_c}^{(1)}(0)$, we count only configurations with backbone $(0, s, 0)$ where $\|s\|_1 = 1$. By using inclusion-exclusion to account for the avoidance between the rib at s and the two ribs at 0, we obtain

$$\Pi_{z_c}^{(1)}(0) \geq 2dz_c^2(g_c^3 - 2g_cQ(s)) = \frac{e}{2d} + o(2d)^{-1}, \quad (6.12)$$

by (1.6) and (6.2). Similarly, by considering the symmetric cases where either the rib at 0 contains $\{0, s\}$ or the rib at s contains $\{0, s\}$, we obtain

$$\Pi_{z_c}^{(1)}(s) \geq 2z_c^2(g_c^3 - g_cQ(s)), \quad (6.13)$$

and hence

$$\sum_{s: \|s\|_1=1} \Pi_{z_c}^{(1)}(s) \geq \frac{2e}{2d} + o(2d)^{-1}. \quad (6.14)$$

Altogether, this gives

$$\hat{\Pi}_{z_c}^{(1)} \geq \frac{3e}{2d} + o(2d)^{-1}. \quad (6.15)$$

For the upper bound, excepting the configurations which contributed the leading behaviour to the lower bound, the remaining configurations that contribute to $\hat{\Pi}_{z_c}^{(1)}$ all contain three paths of total length at least 4, and hence are bounded above by $S_{z_c}^{(4,3)} \leq O(2d)^{-2}$. This completes the proof. \square

Lemma 6.3. *For lattice trees or lattice animals,*

$$g_c = e \left[1 + \frac{\frac{3}{2}}{2d} \right] + o(2d)^{-1}, \quad (6.16)$$

$$z_c = e^{-1} \left[\frac{1}{2d} + \frac{\frac{3}{2}}{(2d)^2} \right] + o(2d)^{-2}. \quad (6.17)$$

Proof. We begin by noting that $g_o(z_c) \leq \hat{\Pi}_{z_c}^{(a,0)}$, and we have already observed in (6.9) that $\hat{\Pi}_{z_c}^{(a,0)}$ is of order $O(2d)^{-2}$. Next, we combine the identity $2dr_c = 1 - 2dz_c\hat{\Pi}_{z_c} - 2dz_cG_{z_c}(s)$ of (3.5) with (1.6) and Lemmas 6.1–6.2 to obtain

$$2dr_c = 1 + \frac{3}{2d} - \frac{1}{2d} + o(2d)^{-1} = 1 + \frac{2}{2d} + o(2d)^{-1}. \quad (6.18)$$

Then (6.16) follows immediately after substitution of (6.18) into the right-hand side of the identity for g_c in Theorem 3.1. Finally, (6.17) follows from substitution of (6.16) and the formula for $\hat{\Pi}_{z_c}$ of Lemma 6.2 into (2.4). \square

7 Third term

We now complete the proof of Theorem 1.1. To do this, we first extend the estimates on $G_{z_c}(s)$ and $\hat{\Pi}_{z_c}$ obtained in Lemmas 6.1–6.2. With these extensions, we then extend the estimate on g_c of Lemma 6.3, and finally combine these results with (2.4) to extend the estimate on z_c and thereby complete the proof of Theorem 1.1. To begin, we insert the formulas of Lemma 6.3 into the formula for $Q(s)$ of Lemma 8.3, to obtain

$$Q(s) = 2z_cg_c^3 - \frac{e^2}{(2d)^2} + o(2d)^{-2} = e^2 \left[\frac{2}{2d} + \frac{11}{(2d)^2} \right] + o(2d)^{-2}. \quad (7.1)$$

The estimate we need for $G_{z_c}(s)$ was stated earlier as Theorem 2.1, which for convenience we restate as follows.

Theorem 7.1. *For lattice trees or lattice animals, and for a neighbour s of the origin,*

$$G_{z_c}(s) = e \left[\frac{1}{2d} + \frac{\frac{7}{2}}{(2d)^2} \right] + o(2d)^{-2}. \quad (7.2)$$

Proof. It follows from Lemma 5.1 that $G_{z_c}^{(5)}(s) \leq O(2d)^{-5/2}$, so we need only consider clusters in which a path of length 1 or 3 joins the points 0 and s .

For the lower bound, we consider clusters that either contain the bond $\{0, s\}$ with this bond not in a cycle, or that do not contain $\{0, s\}$ but contain one of the $2d - 2$ paths of length 3 from 0 to s with this path not part of a cycle. The first contribution is equal to

$$z_c(g_c^2 - Q(s)) = e \left[\frac{1}{2d} + \frac{\frac{5}{2}}{(2d)^2} \right] + o(2d)^{-2}, \quad (7.3)$$

by (7.1) and Lemma 6.3. With s' a neighbour of the origin that is orthogonal to s , the second contribution is bounded below by

$$\begin{aligned} & (2d - 2)z_c^3 \sum_{\substack{R_0 \ni 0, R_1 \ni s' \\ R_2 \ni s+s', R_3 \ni s}} z_c^{|R_0|+|R_1|+|R_2|+|R_3|} \prod_{0 \leq i < j \leq 3} (1 + \mathcal{U}_{ij}) \\ & \geq (2d - 2)z_c^3 \sum_{\substack{R_0 \ni 0, R_1 \ni s' \\ R_2 \ni s+s', R_3 \ni s}} z_c^{|R_0|+|R_1|+|R_2|+|R_3|} \left(1 + \sum_{0 \leq i < j \leq 3} \mathcal{U}_{ij} \right) \\ & = (2d - 2)z_c^3 (g_c^4 - 4g_c^2Q(s) - 2g_c^2Q(s + s')). \end{aligned} \quad (7.4)$$

Now we apply (1.6), and the fact that $Q(x) = o(1)$ by Lemma 8.3, to see that this last expression is equal to

$$(2d-2)z_c^3(g_c^4 - 4g_c^2Q(s) - 2g_c^2Q(s+s')) = \frac{e}{(2d)^2} + o(2d)^{-2}. \quad (7.5)$$

Combining the above results gives the lower bound

$$G_{z_c}(s) \geq e \left[\frac{1}{2d} + \frac{\frac{7}{2}}{(2d)^2} \right] + o(2d)^{-2}. \quad (7.6)$$

For an upper bound, we again need only consider the cases where there is a path of length 1 or 3 connecting 0 and s . Suppose first that there is a path of length 1. If the bond $\{0, s\}$ is not in a cycle, then the above argument again gives a contribution

$$z_c(g_c^2 - Q(s)) = e \left[\frac{1}{2d} + \frac{\frac{5}{2}}{(2d)^2} \right] + o(2d)^{-2}. \quad (7.7)$$

On the other hand, if $\{0, s\}$ is part of a cycle, then we need only consider the case where this bond is part of a cycle of length 4, because otherwise there is a path from 0 to s of length at least 5. The contribution from animals containing $\{0, s\}$ within a cycle of length 4 is at most $(2d-2)z_c^4g_c^4 = O(2d)^{-3}$, so this is an error term. Thus the upper bound for the case of direct connection agrees with the lower bound. In addition, the contribution when there is a path of length 3 is at most

$$(2d-2)z_c^3g_c^4 = \frac{e}{(2d)^2} + o(2d)^{-2}, \quad (7.8)$$

so here too the upper and lower bounds match, and the proof is complete. \square

Next, we present three lemmas which extract the terms in $\hat{\Pi}_{z_c}^{(N)}$ up to $o(2d)^{-2}$, for $N = 0, 1, 2$. The case $N = 0$ occurs only for lattice animals, and we begin with this case.

Lemma 7.2. *For lattice animals,*

$$\hat{\Pi}_{z_c}^{(a,0)} = \frac{\frac{3}{2}}{(2d)^2} + o(2d)^{-2}, \quad (7.9)$$

$$g_o(z_c^{(a)}) = \frac{\frac{1}{2}}{(2d)^2} + o(2d)^{-2}. \quad (7.10)$$

Proof. According to its definition in (4.14),

$$\hat{\Pi}_{z_c}^{(a,0)} = \sum_{x \neq 0} \sum_{R \in \mathcal{D}_{0,x}} z_c^{|R|}. \quad (7.11)$$

The smallest contribution to the right-hand side arises when R is a unit square containing 0, with x a nonzero vertex on the square. Therefore, since there are $\frac{1}{2}(2d)(2d-2)$ such squares and 3 nonzero vertices in each one,

$$\hat{\Pi}_{z_c}^{(a,0)} \leq 3\frac{1}{2}(2d)(2d-2)z_c^4g_c^4 + S^{(6,2)} = \frac{\frac{3}{2}}{(2d)^2} + o(2d)^{-2}, \quad (7.12)$$

where we used (1.6) and Lemma 5.1 in the last equality. For a lower bound, we count only the contributions with $0, x$ in a cycle of length 4, and use inclusion exclusion for the branches emanating from the unit square, to obtain

$$\hat{\Pi}_{z_c}^{(a,0)} \geq 3 \frac{1}{2} (2d)(2d-2) z_c^4 [g_c^4 - 4g_c^2 Q(s_1) - 2g_c^2 Q(s+s')] = \frac{\frac{3}{2}}{(2d)^2} + o(2d)^{-2}, \quad (7.13)$$

where we have used (1.6) together with the fact that $Q(x) = o(1)$ by Lemma 8.3. This proves (7.9).

A similar argument gives (7.10), with the factor 3 missing due to the fact that there is no sum over x in g_o . \square

Lemma 7.3. *For lattice trees or lattice animals,*

$$\hat{\Pi}_{z_c}^{(1)} = e \left[\frac{3}{2d} + \frac{\frac{49}{2}}{(2d)^2} \right] + o(2d)^{-2}. \quad (7.14)$$

Proof. We give the proof only for the case of lattice trees. With minor changes, the arguments presented here also lead to a proof for lattice animals.

By definition,

$$\hat{\Pi}_{z_c}^{(1)} = \sum_{x \in \mathbb{Z}^d} \Pi_{z_c}^{(1)}(x). \quad (7.15)$$

Contributions from $x \neq 0, s, s+s'$, where s, s' are orthogonal neighbours of the origin, are bounded above by $S^{(6,3)} = O(2d)^{-3}$ and need not be considered further. By symmetry, we therefore have

$$\hat{\Pi}_{z_c}^{(1)} = \Pi_{z_c}^{(1)}(0) + 2d \Pi_{z_c}^{(1)}(s) + \frac{2d(2d-2)}{2} \Pi_{z_c}^{(1)}(s+s') + O(2d)^{-3}. \quad (7.16)$$

Consider $\Pi_{z_c}^{(1)}(s+s')$. The shortest backbones have length 2 and there are 2 of these. The shortest allowed rib intersections complete the unit square and there are 3 of these corresponding to the 3 possible nonzero intersection points for the ribs at 0 and $s+s'$. Thus we obtain

$$\Pi_{z_c}^{(1)}(s+s') \leq 3 \cdot 2z_c^4 g_c^5 + S^{(6,3)} + O(2d)^{-3} \leq \frac{6e}{(2d)^4} + O(2d)^{-3}. \quad (7.17)$$

Arguments like those we have been using previously can be used to verify that the first term on the right-hand side is also the leading behaviour of a lower bound, and hence

$$\Pi_{z_c}^{(1)}(s+s') = \frac{6e}{(2d)^4} + o(2d)^{-2}. \quad (7.18)$$

This shows that

$$\hat{\Pi}_{z_c}^{(1)} = \Pi_{z_c}^{(1)}(0) + 2d \Pi_{z_c}^{(1)}(s) + \frac{3e}{(2d)^2} + o(2d)^{-2}. \quad (7.19)$$

Consider $\Pi_{z_c}^{(1)}(s)$. We need only consider the contributions due to rib intersections which together with the backbone form a double bond or a unit square, because the remaining contributions

are bounded by $S^{(6,3)} = O(2d)^{-3}$. These backbones have length 1 or 3, respectively. Thus we obtain (the first term is due to the length-1 backbone and the second to the length-3 backbone)

$$\begin{aligned}\Pi_{z_c}^{(1)}(s) &\leq z_c Q(s) + (2d-2)z_c^3 g_c^2 Q(s) + O(2d)^{-3} \\ &= e \left[\frac{2}{(2d)^2} + \frac{16}{(2d)^3} \right] + o(2d)^{-2},\end{aligned}\tag{7.20}$$

by (1.6) and (7.1). It is routine to prove a matching lower bound, yielding

$$2d\Pi_{z_c}^{(1)}(s) = e \left[\frac{2}{2d} + \frac{16}{(2d)^2} \right] + o(2d)^{-2},\tag{7.21}$$

and hence

$$\hat{\Pi}_{z_c}^{(1)} = \Pi_{z_c}^{(1)}(0) + e \left[\frac{2}{2d} + \frac{19}{(2d)^2} \right] + o(2d)^{-2}.\tag{7.22}$$

Finally, we consider the contributions to $\Pi_{z_c}^{(1)}(0)$ due to backbones of length 2 and 4, which we denote as $\Pi_{z_c}^{(1,2)}(0)$ and $\Pi_{z_c}^{(1,4)}(0)$ respectively. First,

$$\Pi_{z_c}^{(1,4)}(0) \leq 2d(2d-2)z_c^4 g_c^5 + S^{(6,1)} = \frac{e}{(2d)^2} + O(2d)^{-3},\tag{7.23}$$

and a routine matching lower bound gives

$$\Pi_{z_c}^{(1,4)}(0) = \frac{e}{(2d)^2} + O(2d)^{-3}.\tag{7.24}$$

Next,

$$\begin{aligned}\Pi_{z_c}^{(1,2)}(0) &= 2dz_c^2 \sum_{\substack{R_0 \ni 0, R_1 \ni s \\ R_2 \ni 0}} z_c^{|R_0|+|R_1|+|R_2|} (1 + \mathcal{U}_{01} + \mathcal{U}_{12} + \mathcal{U}_{01}\mathcal{U}_{12}) \\ &= 2dz_c^2 \left[g_c^3 - 2g_c Q(s) + \sum_{\substack{R_0 \ni 0, R_1 \ni s \\ R_2 \ni 0}} z_c^{|R_0|+|R_1|+|R_2|} \mathcal{U}_{01}\mathcal{U}_{12} \right] \\ &= e \left[\frac{1}{2d} + \frac{\frac{7}{2}}{(2d)^2} \right] + o(2d)^{-2} + 2dz_c^2 \left[\frac{e^3}{2d} + o(2d)^{-1} \right],\end{aligned}\tag{7.25}$$

where we used Lemma 6.3 and Lemmas 8.3–8.4 in the last equality. With (1.6), this gives

$$\Pi_{z_c}^{(1,2)}(0) = \frac{e}{2d} + \frac{\frac{9}{2}e}{(2d)^2} + o(2d)^{-2}.\tag{7.26}$$

Thus we obtain

$$\Pi_{z_c}^{(1)}(0) = e \left[\frac{1}{2d} + \frac{\frac{11}{2}}{(2d)^2} \right] + o(2d)^{-2}.\tag{7.27}$$

Altogether, we have

$$\hat{\Pi}_{z_c}^{(1)} = e \left[\frac{3}{2d} + \frac{\frac{49}{2}}{(2d)^2} \right] + o(2d)^{-2},\tag{7.28}$$

which proves (7.14). \square

Lemma 7.4. *For lattice trees or lattice animals,*

$$\hat{\Pi}_{z_c}^{(2)} = \frac{11e}{(2d)^2} + o(2d)^{-2}. \quad (7.29)$$

Proof. We defer the proof to Lemma 8.5. \square

For convenience, we now restate Theorem 2.2, supplemented with an asymptotic formula for $g_o(z_c^{(a)})$. Note that the factor e is not present for $g_o(z_c^{(a)})$. It is in Theorem 7.5 that we first see a difference between lattice trees and lattice animals.

Theorem 7.5. *For lattice trees or lattice animals,*

$$\hat{\Pi}_{z_c} = e \left[-\frac{3}{2d} - \frac{\frac{27}{2} - \mathbb{1}_a \frac{3}{2} e^{-1}}{(2d)^2} \right] + o(2d)^{-2}, \quad (7.30)$$

$$g_o(z_c^{(a)}) = \frac{\mathbb{1}_a \frac{1}{2}}{(2d)^2} + o(2d)^{-2}. \quad (7.31)$$

Proof. This follows immediately from Lemmas 7.2–7.4, together with the bounds on $\hat{\Pi}_{z_c}^{(N)}$ for $N > 2$ given by (4.6) and (4.13). \square

The next theorem restates Theorem 1.1, and completes its proof (apart from the technical lemmas of Section 8).

Theorem 7.6. *For lattice trees or lattice animals,*

$$g_c = e \left[1 + \frac{\frac{3}{2}}{2d} + \frac{\frac{263}{24} - \mathbb{1}_a e^{-1}}{(2d)^2} \right] + o(2d)^{-2}, \quad (7.32)$$

$$z_c = e^{-1} \left[\frac{1}{2d} + \frac{\frac{3}{2}}{(2d)^2} + \frac{\frac{115}{24} - \mathbb{1}_a \frac{1}{2} e^{-1}}{(2d)^3} \right] + o(2d)^{-3}. \quad (7.33)$$

Proof. By (3.6) and (7.31),

$$g_c = e^{2dr_c} \left[1 - \frac{1}{2}(2d)r_c^2 + \frac{1}{8}(2d)^2 r_c^4 - \frac{\frac{7}{6}}{(2d)^2} \right] + \frac{\mathbb{1}_a \frac{1}{2}}{(2d)^2} + o(2d)^{-2}. \quad (7.34)$$

The identity (3.5), together with the results for z_c , $\hat{\Pi}_{z_c}$ and $G_{z_c}(s)$ in Lemma 6.3 and Theorems 7.1 and 7.5, imply that

$$2dr_c = 1 - 2dz_c \hat{\Pi}_{z_c} - 2dz_c G_{z_c}(s) = 1 + \frac{2}{2d} + \frac{13 - \mathbb{1}_a \frac{3}{2} e^{-1}}{(2d)^2} + o(2d)^{-2}. \quad (7.35)$$

Substitution of (7.35) into (7.34) gives (7.32). Finally, (7.33) follows immediately by substituting (7.30) and (7.32) into (2.4). \square

8 Cluster intersection estimates

The analysis in Sections 3, 6, and 7 relies heavily on the estimates in this section, which make use only of the leading behaviour (1.6) proved in [26], and of Lemma 5.1. In particular, we will use the fact that

$$2dr_c = 2dz_c g_c - 2dz_c G_{z_c}(s) = 1 + o(1), \quad (8.1)$$

by (3.4) and (1.6), together with the fact that $G_{z_c}(s) = S_{z_c}^{(1,1)} = o(1)$ by Lemma 5.1. Section 8.1 provides the estimates needed for the proof of Theorem 3.1. Section 8.2 provides estimates needed in Sections 6–7.

8.1 Estimates for one-point function

We prove two lemmas which provide estimates needed in the proof of Theorem 3.1. The first gives estimates for $Z^{(i)}$ ($i = 1, 2, 3$) defined in (3.23)–(3.25), as well as for Z' , Z'' defined by

$$Z'(z) = \sum_{s_1, s_2, s_3, s_4 \in \mathcal{E}} \sum_{S_1 \ni s_1} \sum_{S_2 \ni s_2} \sum_{S_3 \ni s_3} \sum_{S_4 \ni s_4} z^{|S_1|+|S_2|+|S_3|+|S_4|} (-\mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{14}), \quad (8.2)$$

$$Z''(z) = \sum_{s_1, s_2, s_3, s_4 \in \mathcal{E}} \sum_{S_1 \ni s_1} \sum_{S_2 \ni s_2} \sum_{S_3 \ni s_3} \sum_{S_4 \ni s_4} z^{|S_1|+|S_2|+|S_3|+|S_4|} (-\mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{24}). \quad (8.3)$$

We use the subscript c to denote quantities evaluated at z_c .

Lemma 8.1. *For lattice trees or lattice animals,*

$$Z_c^{(1)} = 2dr_c^2 + \frac{3}{(2d)^2} + o(2d)^{-2}, \quad (8.4)$$

$$Z_c^{(2)} = \frac{1}{(2d)^2} + o(2d)^{-2}, \quad (8.5)$$

$$Z_c^{(3)} = \frac{1}{(2d)^2} + o(2d)^{-2}, \quad (8.6)$$

$$Z'_c = Z''_c = O(2d)^{-3}. \quad (8.7)$$

Proof. We consider the four equations in turn.

Proof of (8.4). According to its definition in (3.23),

$$Z_c^{(1)} = \sum_{s_1, s_2 \in \mathcal{E}} \sum_{S_1 \ni s_1} \sum_{S_2 \ni s_2} z_c^{|S_1|+|S_2|} (-\mathcal{V}_{12}). \quad (8.8)$$

We distinguish the two possibilities $|\{s_1, s_2\}| = 1, 2$ for the vertices s_1, s_2 , i.e., we distinguish whether the two vertices are equal or not. If $s_1 = s_2$ then automatically $-\mathcal{V}_{12} = 1$ because both clusters contain s_1 , and this contribution gives exactly $2dr_c^2$.

For $s_1 \neq s_2$, we consider separately the cases where s_1 and s_2 are parallel and perpendicular. This contributes

$$2d \sum_{\substack{S_1 \ni s_1 \\ S_2 \ni -s_1}} z_c^{|S_1|+|S_2|} (-\mathcal{V}_{12}) + 2d(2d-2) \sum_{\substack{S_1 \ni s_1 \\ S_2 \ni s_2}} z_c^{|S_1|+|S_2|} (-\mathcal{V}_{12}). \quad (8.9)$$

For the first term, at least six steps are required for an intersection of S_1 and S_2 , so this term is bounded above by $S^{(6,2)} = O(2d)^{-3}$, by Lemma 5.1. The leading behaviour of the second term is $3(2d)(2d-2)z_c^4 g(z_c)^4$ (due to a square containing 0, s_1 and s_2 , and where the factor 3 takes into account the three nonzero vertices of the square at which S_1, S_2 might intersect). The remaining contributions are bounded above by $S^{(6,2)} = O(2d)^{-3}$. It is not difficult to prove a corresponding lower bound, to conclude that (8.9) equals

$$3(2d)^2 z_c^4 g_c^4 + o(2d)^{-2} = \frac{3}{(2d)^2} + o(2d)^{-2}, \quad (8.10)$$

where we used (1.6) for the last equality. When combined with the contribution from $s_1 = s_2$, this completes the proof of (8.4).

Proof of (8.5). By definition

$$Z_c^{(2)} = \sum_{s_1, s_2, s_3 \in \mathcal{E}} \sum_{S_1 \ni s_1} \sum_{S_2 \ni s_2} \sum_{S_3 \ni s_3} z_c^{|S_1|+|S_2|+|S_3|} \mathcal{V}_{12} \mathcal{V}_{13}. \quad (8.11)$$

We distinguish the three possibilities $|\{s_1, s_2, s_3\}| = 1, 2, 3$ for the vertices s_1, s_2 and s_3 .

If $|\{s_1, s_2, s_3\}| = 1$, then automatically $\mathcal{V}_{12} \mathcal{V}_{13} = 1$. In this case, using (8.1), we find that the contribution to $Z_c^{(2)}$ becomes simply

$$2dr_c^3 = \frac{1}{(2d)^2} + o(2d)^{-2}. \quad (8.12)$$

If $|\{s_1, s_2, s_3\}| = 2$, then we consider the case $s_1 \neq s_2 = s_3$ (the other cases can be handled with similar arguments). In this case, we use $|\mathcal{V}_{13}| \leq 1$ and perform the sum over S_3 to obtain a factor r_c . The remaining sum is the case $s_1 \neq s_2$ studied in the bound on $Z_c^{(1)}$ and shown above in (8.10) to be $O(2d)^{-2}$. Thus this contribution is an error term, since $r_c = O(2d)^{-1}$ by (8.1).

If $|\{s_1, s_2, s_3\}| = 3$, then all three vertices are different. At least seven bonds are required to obtain $\mathcal{V}_{12} \mathcal{V}_{13} = 1$ in this case. As depicted in Figure 6, this contribution is bounded above by $\sum_{i+j=7} S^{(i,2)} S^{(j,3)}$ and hence is $O(2d)^{-7/2}$, another error term. This completes the proof of (8.5).

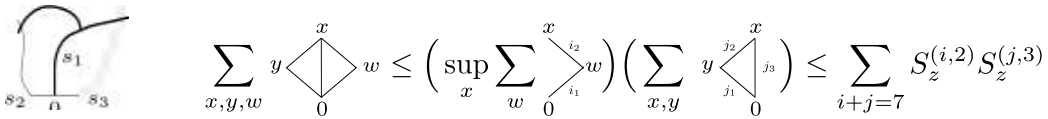


Figure 6: Intersections required for the case $|\{s_1, s_2, s_3\}| = 3$ of $Z^{(2)}$, with corresponding bound. The paths have lengths at least i_k and j_l , with $i_1 + i_2 = i$, $j_1 + j_2 + j_3 = j$, and $i + j = 7$.

Proof of (8.6). By definition,

$$Z_c^{(3)} = \sum_{s_1, s_2, s_3 \in \mathcal{E}} \sum_{S_1 \ni s_1} \sum_{S_2 \ni s_2} \sum_{S_3 \ni s_3} z_c^{|S_1|+|S_2|+|S_3|} (-\mathcal{V}_{12} \mathcal{V}_{13} \mathcal{V}_{23}). \quad (8.13)$$

If $|\{s_1, s_2, s_3\}| = 1$, then automatically $-\mathcal{V}_{12} \mathcal{V}_{13} \mathcal{V}_{23} = 1$ and hence

$$Z_c^{(3)} \geq 2dr_c^3 = \frac{1}{(2d)^2} + o(2d)^{-2}. \quad (8.14)$$

On the other hand the inequality $-\mathcal{V}_{23} \leq 1$, together with (8.5), shows that

$$Z_c^{(3)} \leq Z_c^{(2)} = \frac{1}{(2d)^2} + o(2d)^{-2}. \quad (8.15)$$

This completes the proof of (8.6).

Proof of (8.7). We prove that Z'_c and Z''_c are $O(2d)^{-3}$. For each, we distinguish the four possibilities $|\{s_1, s_2, s_3, s_4\}| = 1, 2, 3, 4$.

If $|\{s_1, s_2, s_3, s_4\}| = 1$, the products $-\mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{14} = 1$ and $-\mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{24}$ are equal to 1, and the sums in (8.2) and (8.3) reduce to $2dr_c^4$, which is $O(2d)^{-3}$ by (8.1).

If $|\{s_1, s_2, s_3, s_4\}| = 2$, we decompose the products $-\mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{14}$ and $-\mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{24}$ into a factor that involves the two different vertices, and the remaining two factors. We bound these last two factors by 1, so their corresponding sums are bounded by $r_c^2 = O(2d)^{-2}$. The remaining sums are equal to (8.9), which by (8.10) is of order $O(2d)^{-2}$.

If $|\{s_1, s_2, s_3, s_4\}| = 3$, we decompose $-\mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{14}$ and $-\mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{24}$ into two factors involving the three distinct vertices, and one remaining factor. We bound the latter factor by 1, and the corresponding sum becomes $r_c = O(2d)^{-1}$. The remaining sums are bounded by $Z_c^{(2)}$ for the case $-\mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{14}$, and by $Z_c^{(2)}$ or $(Z_c^{(1)})^2$ for the case $-\mathcal{V}_{12}\mathcal{V}_{13}\mathcal{V}_{24}$. The overall contribution in both cases is therefore $O(2d)^{-3}$, by (8.4) and (8.5),

If $|\{s_1, s_2, s_3, s_4\}| = 4$, an example of the required intersections for Z' is depicted in Figure 7. By taking into account all possibilities for Z' and Z'' , we draw the crude conclusion that at least six bonds are needed to achieve the required intersections, and this leads to an upper bound of the form $\sum_{n_1+n_2+n_3=6} O(S^{(n_1,M)} S^{(n_2,M)} S^{(n_3,M)})$, for a fixed value of M , and hence is $O(2d)^{-3}$.

This completes the proof of (8.7) and of the lemma. \square

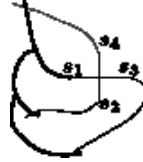


Figure 7: Example of intersections for the case $|\{s_1, s_2, s_3, s_4\}| = 4$ of Z' .

Lemma 8.2. *For lattice trees or lattice animals,*

$$\Gamma_c^{(3,n)} = O(2d)^{-3} \quad (n = 4, 5, 6), \quad (8.16)$$

$$\tilde{\Gamma}_c^{(4)} = O(2d)^{-3}. \quad (8.17)$$

Proof. We consider the two equations in turn.

Proof of (8.16). First we consider $\Gamma^{(3,4)}$, and will show that

$$\Gamma^{(3,4)}(z) = \Gamma^{(0)}(z) \left(\frac{4}{4!} Z'(z) + \frac{12}{4!} Z''(z) \right). \quad (8.18)$$

This is sufficient, by (8.7) together with the fact that $\Gamma_c^{(0)} = e^{2dr_c} = O(1)$ by (3.14) and (8.1). To prove (8.18), we are considering the case where the set of labels $\{i, j, k, l, p, q\}$ in (3.21) has cardinality 4, and we may assume the labels are 1, 2, 3, 4. We find 16 possible arrangements for the labels, which can be reduced to the two cases:

- (i) Three labels are equal and the other three are different from the first ones and among them, e.g., $i = k = p = 1$, $j = 2$, $l = 3$ and $q = 4$. There are 4 arrangements of this type.
- (ii) There are two pairs of equal labels and a pair of distinct labels, e.g., $i = k = 1$, $j = p = 2$, $l = 3$ and $q = 4$. There are 12 arrangements of this type.

Interchanging the sums in (3.21) and using symmetry, as in the proof of Lemma 3.2, gives (8.18).

For $\Gamma_c^{(3,5)}$, one of the factors \mathcal{V}_{ij} has labels that do not repeat, and the other two factors share one of the labels. The sums over the s and S with the two non-repeating labels yield $Z_c^{(1)} = O(2d)^{-1}$. The sums over the remaining labels are bounded above by $Z_c^{(2)} = O(2d)^{-2}$. It is then straightforward to verify that $\Gamma_c^{(3,5)} \leq O(2d)^{-3}$.

For $\Gamma_c^{(3,6)}$, the six sums over s give $(Z_c^{(1)})^3 = O(2d)^{-3}$, and this leads to $\Gamma_c^{(3,6)} \leq O(2d)^{-3}$. This completes the proof of (8.16).

Proof of (8.17). We use the bound $|\mathcal{I}_{rs}| \leq 1$ in (3.22) to obtain

$$|\tilde{\Gamma}_c^{(4)}| \leq \sum_{m=4}^{\infty} \frac{1}{m!} \sum_{s_1, \dots, s_m \in \mathcal{E}} \sum_{S_1 \ni s_1} z_c^{|S_1|} \dots \sum_{S_m \ni s_m} z_c^{|S_m|} \sum_{1 \leq i < j \leq m} \sum_{(k,l) \in A_{ij}} \sum_{(p,q) \in A_{kl}} \sum_{(r,s) \in A_{pq}} \mathcal{V}_{ij} \mathcal{V}_{kl} \mathcal{V}_{pq} \mathcal{V}_{rs}. \quad (8.19)$$

We denote the cardinality of the label set by $n = |\{i, j, k, l, p, q, r, s\}|$, so $n \in \{4, 5, 6, 7, 8\}$, and exchange the sums over vertices and labels. As in (8.18), this allows us to rewrite the upper bound of (8.19) in the form

$$|\tilde{\Gamma}_c^{(4)}| \leq \Gamma_c^{(0)} \sum_{n=4}^8 \sum_i \alpha_{n,i} Z_c^{(4,n,i)}, \quad (8.20)$$

where the sum over i is a finite sum, the $\alpha_{n,i}$ are constants whose values are immaterial, and each $Z_c^{(4,n,i)}$ is of the form

$$Z_c^{(4,n,i)} = \sum_{s_1, \dots, s_n \in \mathcal{E}} \sum_{S_1 \ni s_1} z_c^{|S_1|} \dots \sum_{S_n \ni s_n} z_c^{|S_n|} \mathcal{V}^{(n,i)}, \quad (8.21)$$

with $\mathcal{V}^{(n,i)}$ a product of 4 factors of \mathcal{V}_{ab} having n distinct labels in all. Since $\Gamma_c^{(0)} = O(1)$ (as observed below (8.18)), it suffices to show that each $Z_c^{(4,n,i)}$ is $O(2d)^{-3}$.

For $n = 4, 5$ or 6 , we substitute one of the factors in $|\mathcal{V}_{ij} \mathcal{V}_{kl} \mathcal{V}_{pq} \mathcal{V}_{rs}|$ by 1, with the restriction that the remaining three factors have at least four different labels. The sums involving the replaced factor yield 1 or $2dr_c$ or $(2dr_c)^2$ depending on whether this factor has 0, 1 or 2 distinct labels from the remaining three factors; all three cases are $O(1)$ by (8.1). The sums involving the other three factors reduce to the cases $\Gamma_c^{(3,4)}$, $\Gamma_c^{(3,5)}$ or $\Gamma_c^{(3,6)}$ which by (8.16) are $O(2d)^{-3}$.

For $n = 7$ or 8 , we consider three factors in the product $|\mathcal{V}_{ij} \mathcal{V}_{kl} \mathcal{V}_{pq} \mathcal{V}_{rs}|$ that have six different labels and bound the fourth factor by 1. The sums involving the fourth factor yield $2dr_c$ and

$(2dr_c)^2$ for $n = 7$ and $n = 8$, respectively. By (8.1), in both cases the contribution is $O(1)$. By the bound on $\Gamma_c^{(3,6)}$ of (8.16), the sums involving the six distinct labels is $O(2d)^{-3}$. This completes the proof of (8.17) and of the lemma. \square

8.2 Estimates for lace expansion

We now prove three lemmas that were used in Sections 6–7. Recall from (6.1) the definition

$$Q(x) = \sum_{C_0 \ni 0} \sum_{C_x \ni x} z_c^{|C_0|+|C_x|} \mathbb{1}_{C_0 \cap C_x \neq \emptyset}. \quad (8.22)$$

The following lemma gives a good estimate for $Q(s)$ when $\|s\|_1 = 1$, and gives a crude (but sufficient) estimate for $\|x\|_1 > 1$.

Lemma 8.3. *For lattice trees or lattice animals, and for a neighbour s of the origin,*

$$Q(s) = 2z_c g_c^3 - \frac{e^2}{(2d)^2} + o(2d)^{-2} = \frac{2e^2}{2d} + o(2d)^{-1}. \quad (8.23)$$

In addition, for any x , $Q(x) \leq O(2d)^{-\frac{1}{2}\|x\|_1}$.

Proof. In (8.22), the clusters C_0 and C_x only contribute to the sum in $Q(x)$ if they have a vertex in common, say y . There is a path connecting 0 and y contained in C_0 , and a path connecting y and x contained in C_1 , and we can choose these paths to intersect only at y . We denote the paths by ω^0 and ω^1 , respectively. The union of ω^0 and ω^1 forms a path connecting 0 to x and passing through y , which we call ω . It has length at least $\|x\|_1$, and this leads to the upper bound $Q(x) \leq S^{(\|x\|_1, 2)}$. Together with Lemma 5.1, this proves that $Q(x) \leq O(2d)^{-\frac{1}{2}\|x\|_1}$.

It remains to prove the first equality of (8.23), as the second equality then follows immediately from (1.6). We write $Q^n(s)$ to refer to the contribution to $Q(s)$ due to configurations where there exists such a path ω of length n and no shorter path. Since

$$Q^{\geq 5}(s) \leq S_{z_c}^{(5,2)} \leq O(2d)^{-5/2}, \quad (8.24)$$

we can restrict attention to Q^n for $n \leq 4$. For the case of lattice animals, the contributions in which C_0 or C_1 has a cycle containing both 0 and s is easily seen to be $o(2d)^{-2}$. Therefore, for the rest of the proof we assume that each of C_0 and C_1 does not have a cycle that contains both 0 and s .

For Q^3 , we have $\omega = (0, s', s' + s, s)$ for some s' perpendicular to s . There are $2d - 2$ such paths and each of them has four possibilities for y . If we treat the clusters attached to the vertices in ω^0 and ω^1 as five independent clusters, we obtain the upper bound

$$Q^3(s) \leq 4(2d - 2)z_c^3 g_c^5 = \frac{4e^2}{(2d)^2} + o(2d)^{-2}, \quad (8.25)$$

with the last equality due to (1.6). For a lower bound, we use inclusion-exclusion and subtract from the upper bound the contribution when there are pairwise intersections among the ribs that belong to the same path, either ω^0 or ω^1 . This gives

$$Q^3(s) \geq 4(2d - 2)z_c^3 g_c^3 [g_c^2 - 4Q(s) - 2Q(s' + s)] = \frac{4e^2}{(2d)^2} + o(2d)^{-2} \quad (8.26)$$

(subtraction of $Q(s)$ in the middle expression also accounts for configurations which are counted by $Q^1(s)$ rather than $Q^3(s)$). We conclude that

$$Q^3(s) = \frac{4e^2}{(2d)^2} + o(2d)^{-2}. \quad (8.27)$$

For Q^1 , the path ω is given by $\omega = (0, s)$. This means that the bond $\{0, s\}$ is contained in either C_0 or C_1 , say in C_0 . In this case, C_0 consists of the edge $\{0, s\}$ and two nonintersecting subclusters, C_0^* and C_1^* , the first one attached at 0 and the second at s . Let $\mathcal{U}_{01}^* = -1$ if the subclusters C_0^* and C_1^* have a common vertex, and 0 otherwise. Exchanging the roles of C_0 and C_1 , and subtracting the contribution due to the event in which both clusters C_0 and C_1 contain the bond $\{0, s\}$, yields

$$\begin{aligned} Q^1(s) &= 2z_c g_c \sum_{C_0^* \ni 0} \sum_{C_1^* \ni s} z_c^{|C_0^*|+|C_1^*|} (1 + \mathcal{U}_{01}^*) - z_c^2 \left[\sum_{C_0^* \ni 0} \sum_{C_1^* \ni s} z_c^{|C_0^*|+|C_1^*|} (1 + \mathcal{U}_{01}^*) \right]^2 \\ &= 2z_c g_c^3 - 2z_c g_c Q(s) - z_c^2 [g_c^2 - Q(s)]^2. \end{aligned} \quad (8.28)$$

Since $z_c^2 Q(s) = o(2d)^{-2}$, together with the contributions analysed previously this gives

$$Q(s) = 2z_c g_c^3 - 2z_c g_c Q(s) - z_c^2 g_c^4 + \frac{4e^2}{(2d)^2} + o(2d)^{-2}. \quad (8.29)$$

We conclude from this that

$$(1 + 2z_c g_c)Q(s) = 2z_c g_c^3 + \frac{3e^2}{(2d)^2} + o(2d)^{-2}. \quad (8.30)$$

The factor multiplying $Q(s)$ is equal to $1 + 2(2d)^{-1} + o(2d)^{-1}$, so $Q(s)$ is equivalent to multiply the right-hand side by $1 - 2(2d)^{-1} + o(2d)^{-1}$. This yields the first equality of (8.23) and completes the proof. \square

The next lemma is applied in Lemmas 7.3 and 8.5. For a neighbour s of the origin, we define

$$Q^*(s) = \sum_{C_0 \ni 0} \sum_{C_1 \ni s} \sum_{C_2 \ni 0} z_c^{|R_0|+|R_1|+|R_2|} \mathcal{U}_{01} \mathcal{U}_{12}. \quad (8.31)$$

Lemma 8.4. *For lattice trees or lattice animals, and for a neighbour s of the origin,*

$$Q^*(s) = \frac{e^3}{2d} + o(2d)^{-1}, \quad (8.32)$$

Proof. It is straightforward to verify that the contribution when C_1 contains a cycle containing 0 and s produces an error term, so we assume that there is no such cycle. If C_1 contains the bond $(0, s)$, then $\mathcal{U}_{01} \mathcal{U}_{12} = 1$. In this case, we can regard C_1 as consisting of the edge $(0, s)$ and two

non-intersecting clusters C_1^0 and C_1^1 attached at 0 and s , respectively. Let $\mathcal{U}_{01}^* = -1$ if C_1^0 and C_1^1 have a common vertex, and 0 otherwise. We obtain

$$Q^*(s) = z_c g_c^2 \sum_{R_1^0 \ni 0, R_1^1 \ni e_1} z_c^{|R_1^0|+|R_1^1|} (1 + \mathcal{U}_{01}^*) + \sum_{\substack{R_0, R_2 \ni 0 \\ R_1 \ni e_1, R_1 \not\ni (0, e_1)}} z_c^{|R_0|+|R_1|+|R_2|} \mathcal{U}_{01} \mathcal{U}_{12} + o(2d)^{-1}. \quad (8.33)$$

Arguments of the type used several times previously show that the second sum on the right-hand side is $o(2d)^{-1}$. Therefore,

$$Q^*(s) = z_c g_c^4 - z_c g_c^2 Q(s) + o(2d)^{-1} = \frac{e^3}{2d} + o(2d)^{-1}, \quad (8.34)$$

where the second equality is due to (1.6) and Lemma 8.3. \square

Finally, we prove the following lemma, which is a restatement of Lemma 7.4. It provides an important ingredient in the proof of Theorem 7.5.

Lemma 8.5. *For lattice trees or lattice animals,*

$$\hat{\Pi}_{z_c}^{(2)} = \frac{11e}{(2d)^2} + o(2d)^{-2}. \quad (8.35)$$

Proof. We give the proof only for the case of lattice trees. With minor changes, the proof extends to lattice animals. Recall from (4.9) that

$$\hat{\Pi}_{z_c}^{(2)} = \sum_{x \in \mathbb{Z}^d} \Pi_{z_c}^{(2)}(x) = \sum_{x \in \mathbb{Z}^d} \sum_{\substack{\omega \in \mathcal{W}(x) \\ |\omega| \geq 2}} z_c^{|\omega|} \left[\prod_{i=0}^{|\omega|} \sum_{R_i \ni \omega(i)} z_c^{|R_i|} \right] \left[\sum_{L \in \mathcal{L}^{(2)}[0, |\omega|]} \prod_{ij \in L} \mathcal{U}_{ij} \prod_{i'j' \in \mathcal{C}(L)} (1 + \mathcal{U}_{i'j'}) \right], \quad (8.36)$$

where the set of laces is

$$\mathcal{L}^{(2)}[0, |\omega|] = \{ \{0j, j|\omega|\} : 0 < j < |\omega| \} \cup \{ \{0j, i|\omega|\} : 0 < i < j < |\omega| \}, \quad (8.37)$$

and where the set $\mathcal{C}(L)$ compatible with L is defined below (4.9). Let $\Pi_{z_c}^{(2,n)}(x)$ denote the contribution to $\Pi_{z_c}^{(2)}(x)$ due to $|\omega| = n$ on the right-hand side of (8.36). We will show that

$$\hat{\Pi}_{z_c}^{(2,2)} = \frac{5e}{(2d)^2} + o(2d)^{-2}, \quad (8.38)$$

$$\hat{\Pi}_{z_c}^{(2,3)} = \frac{5e}{(2d)^2} + o(2d)^{-2}, \quad (8.39)$$

$$\hat{\Pi}_{z_c}^{(2,4)} = \frac{e}{(2d)^2} + o(2d)^{-2}, \quad (8.40)$$

$$\hat{\Pi}_{z_c}^{(2,>4)} = o(2d)^{-2}, \quad (8.41)$$

which proves (8.35).

Before entering into the details, we recall diagrammatic estimates for lattice trees that have been developed and discussed at length in [12, 13, 30] (for lattice animals the best reference is [12]). These techniques are based on the diagrams in Figure 8, which inspire the upper bound

$$\hat{\Pi}_{z_c}^{(2)} \leq S^{(1,3)} S^{(1,4)}. \quad (8.42)$$

Here the occurrence of $S^{(1,n)}$ on the right-hand side is connected with the fact that each loop in the bounds on diagrams in Figure 8 must consist of at least one bond, while the appearance of 3 and 4 is due to the 7 lines in the adjacent squares, each of which represents a two-point function. When we consider configurations for which it is guaranteed that those two-point functions must take at least k steps in total, the upper bound (8.42) can be improved to an upper bound

$$\sum_{i+j=k} S^{(i,4)} S^{(j,3)} = O(2d)^{-k/2}, \quad (8.43)$$

and once $k = 5$ this is an error term. We will exploit this principle in the following, beginning with (8.41) for its simplest illustration.

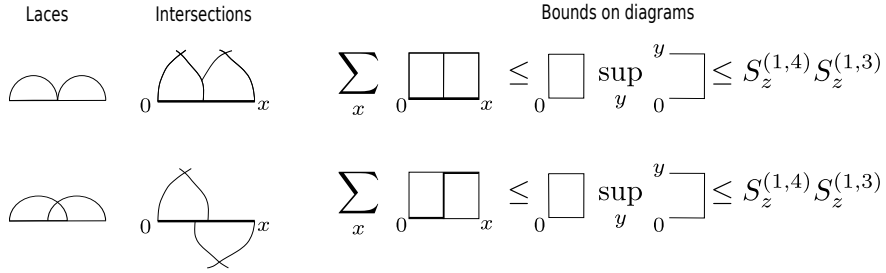


Figure 8: The two generic laces consisting of two bonds, schematic diagrams showing the corresponding rib intersections for a nonzero contribution to $\Pi^{(2)}(x)$, and diagrammatic bounds for the contributions to $\Pi^{(2)}(x)$. Diagram lines corresponding to the backbone joining 0 and x are shown in bold.

Proof of (8.41). When ω has length at least 5, then from (8.43) we immediately obtain

$$\hat{\Pi}_{z_c}^{(2,>4)} \leq \sum_{i+j=5} S^{(i,3)} S^{(j,4)} \leq O(2d)^{-5/2}, \quad (8.44)$$

which gives (8.41).

Proof of (8.38). When $|\omega| = 2$, there is only the lace $L = \{01, 12\}$, and $\mathcal{C}(L) = \emptyset$. Therefore,

$$\hat{\Pi}_{z_c}^{(2,2)} = \sum_{x: \|x\|_1 \in \{0,2\}} \sum_{\substack{\omega \in \mathcal{W}(x) \\ |\omega|=2}} z_c^2 \sum_{\substack{R_0 \ni \omega(0), R_1 \ni \omega(1) \\ R_2 \ni \omega(2)}} z_c^{|R_0|+|R_1|+|R_2|} \mathcal{U}_{01} \mathcal{U}_{12}. \quad (8.45)$$

For $x = 0$, we have $\omega = (0, s, 0)$, where s is a neighbour of the origin, and Lemma 8.4 gives

$$\Pi_{z_c}^{(2,2)}(0) = 2dz_c^2 \left[\frac{e^3}{2d} + o(2d)^{-1} \right] = \frac{e}{(2d)^2} + o(2d)^{-2}. \quad (8.46)$$

When $\|x\|_1 = 2$, one way to achieve $\mathcal{U}_{01}\mathcal{U}_{12} = 1$ is to have either R_0 or R_1 contain the bond $(0, s)$, and either R_1 or R_2 contain the bond (s, x) . To obtain a lower bound from such configurations, we treat the subribs emanating from these bonds as independent and use inclusion-exclusion to subtract the possible intersections among them. This yields

$$\sum_{x:\|x\|_1=2} \Pi_{z_c}^{(2,2)}(x) \geq 4(2d)(2d-1)z_c^4 g_c [g_c^4 - 2Q(s)z_c g_c^2] = \frac{4e}{(2d)^2} + o(2d)^{-2}. \quad (8.47)$$

If $(0, s)$ is not present in R_0 and R_1 , or (s, x) is not present in R_1 and R_2 , then an intersection among the corresponding ribs requires at least four edges (including the step in ω), so as in Figure 8 we obtain for this case the crude upper bound $S^{(4,4)}S^{(1,3)} + S^{(1,4)}S^{(4,3)}$. This implies

$$\sum_{x:\|x\|=2} \Pi_{z_c}^{(2,2)}(x) \leq 4(2d)(2d-1)z_c^4 g_c^5 + S^{(4,4)}S^{(1,3)} + S^{(1,4)}S^{(4,3)} = \frac{4e}{(2d)^2} + o(2d)^{-2}, \quad (8.48)$$

and, with (8.46)–(8.47), this completes the proof of (8.38).

Proof of (8.39). When $|\omega| = 3$, there are three laces: $L = \{01, 13\}$, $L = \{02, 23\}$, $L = \{02, 13\}$.

The laces $L = \{01, 13\}$, $L = \{02, 23\}$. By symmetry, both laces give the same contribution to (8.36), so of these we only study the contribution due to $L = \{01, 13\}$ (with $\mathcal{C}(L) = \{12, 23\}$), which is

$$\sum_{x:\|x\|_1 \in \{1,3\}} \sum_{\substack{\omega \in \mathcal{W}(x) \\ |\omega|=3}} z_c^3 \sum_{\substack{R_0 \ni \omega(0), R_1 \ni \omega(1) \\ R_2 \ni \omega(2), R_3 \ni \omega(3)}} z_c^{|R_0|+|R_1|+|R_2|+|R_3|} \mathcal{U}_{01}\mathcal{U}_{13} (1 + \mathcal{U}_{12}) (1 + \mathcal{U}_{23}). \quad (8.49)$$

Case of $\|x\|_1 = 1$. When $\|x\|_1 = 1$, ω either has the form $\omega = (0, x, y, x)$ for y a neighbour of x (possibly $y = 0$), or $\omega = (0, s, s + y, x)$ for a neighbour s of the origin distinct from x and for $y \in \{-s, x\}$. In the first case, when $\omega = (0, x, y, x)$, we have $\mathcal{U}_{13} = -1$ since $\omega(1) = x = \omega(3)$. Using $(1 + \mathcal{U}_{12})(1 + \mathcal{U}_{23}) \leq 1$, (1.6) and Lemma 8.3, we find that this contribution to (8.49) is bounded above by

$$(2d)^2 z_c^3 g_c^2 Q(x) = \frac{2e}{(2d)^2} + o(2d)^{-2}. \quad (8.50)$$

Also, using $(-\mathcal{U}_{01})(1 + \mathcal{U}_{12})(1 + \mathcal{U}_{23}) \geq (-\mathcal{U}_{01})(1 + \mathcal{U}_{12} + \mathcal{U}_{23})$, this contribution to (8.49) is bounded below by

$$(2d)^2 z_c^3 [g_c^2 Q(x) - O(2d)^{-2} - Q(x)^2] = \frac{2e}{(2d)^2} + o(2d)^{-2}, \quad (8.51)$$

where we omit the straightforward details for the $\mathcal{U}_{01}\mathcal{U}_{12}$ term. This contribution gets counted twice to account also for the lace $L = \{02, 23\}$.

In the second case, when $\omega = (0, s, s + x, x)$, the contribution to (8.49) is bounded above by

$$(2d)^2 z_c^3 g_c^2 \sum_{R_1 \ni s, R_3 \ni x} z_c^{|R_1|+|R_3|} (-\mathcal{U}_{13}) = (2d)^2 z_c^3 g_c^2 Q(x - s) = o(2d)^{-2}, \quad (8.52)$$

where we have employed the straightforward improvement $Q(x-s) \leq O(2d)^{-2}$ to the crude bound of Lemma 8.3, for $\|x-s\|_1 = 2$. Also, when $\omega = (0, s, 0, x)$, it can be checked that due to the factor $(1 + \mathcal{U}_{12})$ at least 6 bonds are required to accomplish the intersections required for $\mathcal{U}_{01}\mathcal{U}_{13} = 1$. Therefore, using the upper bound (8.43), this contribution is at most $O(2d)^{-3}$.

Case of $\|x\|_1 = 3$. When $\|x\|_1 = 3$, it can be checked that the required intersections cannot be accomplished without using at least 5 bonds, and we conclude from the upper bound (8.43) that the total contribution from all such x is at most $O(2d)^{-5/2}$.

The lace $L = \{02, 13\}$. Its contribution to (8.36) is

$$\sum_{x: \|x\|_1 \in \{1, 3\}} \sum_{\substack{\omega \in \mathcal{W}(x) \\ |\omega| = 3}} z_c^3 \sum_{\substack{R_0 \ni \omega(0), \dots, \\ R_3 \ni \omega(3)}} z_c^{|R_0| + \dots + |R_3|} \mathcal{U}_{02} \mathcal{U}_{13} (1 + \mathcal{U}_{01}) (1 + \mathcal{U}_{12}) (1 + \mathcal{U}_{23}). \quad (8.53)$$

When $\|x\|_1 = 1$, either $\omega = (0, x, 0, x)$, or $\omega = (0, s, s+y, x)$ for a neighbour s of the origin distinct from x and for $y \in \{-s, x\}$. In the first case, automatically $\mathcal{U}_{0,2}\mathcal{U}_{13} = 1$ since $\omega(0) = 0 = \omega(2)$ and $\omega(1) = x = \omega(3)$. The contribution to (8.53) is bounded above by

$$2dz_c^3 g_c^4 = \frac{e}{(2d)^2} + o(2d)^{-2}. \quad (8.54)$$

A matching lower bound is given by

$$2dz_c^3 \sum_{\substack{R_0 \ni 0, \dots, \\ R_3 \ni x}} z_c^{|R_0| + \dots + |R_3|} (1 + \mathcal{U}_{01} + \mathcal{U}_{12} + \mathcal{U}_{23}) \geq 2dz_c^3 [g_c^4 - 3g_c^2 Q(x)] = \frac{e}{(2d)^2} + o(2d)^{-2}. \quad (8.55)$$

In the second case, when $\omega = (0, s, s+y, x)$, the contribution to (8.53) is bounded above by

$$2d(2d-1)z_c^3 g_c^2 \sum_{R_1 \ni s, R_3 \ni x} z_c^{|R_1| + |R_3|} (-\mathcal{U}_{13}) \leq 2d(2d-1)z_c^3 g_c^2 Q(x-s) = o(2d)^{-2}. \quad (8.56)$$

If $\|x\|_1 = 3$, the lace $L = \{02, 13\}$ forces an intersection between the ribs R_1 and R_3 , without intersecting R_2 (due to $12, 23 \in \mathcal{C}(L)$). It can be argued that the contribution in this case is $o(2d)^{-2}$.

Proof of (8.40). For $|\omega| = 4$, we first consider the lace $L = \{02, 24\}$ with $x = 0$, which is the only case that contributes. After discussing this case in detail, we will argue that all remaining contribution belong to the error term.

For $L = \{02, 24\}$ and $x = 0$, the significant walks are $\omega = (0, s, 0, s', 0)$ with s, s' neighbours of the origin. There are $(2d)^2$ such walks and they have $\mathcal{U}_{02}\mathcal{U}_{24} = 1$. Treating the five ribs emanating from the walks as independent, we obtain the upper bound

$$(2d)^2 z_c^4 g_c^5 = \frac{e}{(2d)^2} + o(2d)^{-2}, \quad (8.57)$$

and it is straightforward to verify that this is also a lower bound. This gives the formula $e(2d)^{-2} + o(2d)^{-2}$ that we seek for $\hat{\Pi}^{(2,4)}$, so it remains to prove that the remaining terms contribute $o(2d)^{-2}$.

The other walks of length four with $x = 0$ form unit squares containing the origin (the walk $(0, s, s + s', s, 0)$ does not contribute since it has $1 + \mathcal{U}_{13} = 0$). If we bound \mathcal{U}_{02} by 1, and use the fact that there are $O(2d)^2$ such squares, we find that this contribution to $\Pi_{z_c}^{(2,4)}(0)$ is bounded by (with s, s' orthogonal)

$$O(2d)^2 z_c^4 g_c^3 Q(s + s') = o(2d)^{-2}, \quad (8.58)$$

where $Q(s + s')$ takes into account the intersection of R_2 and R_4 forced by \mathcal{U}_{24} .

For the remaining case $x \neq 0$ for $L = \{02, 24\}$, and for all other laces occurring for $|\omega| = 4$, it can be checked that there must be at least one additional bond, besides the backbone, in order to create the intersections for a nonzero contribution. Then (8.43) gives an upper bound

$$\sum_{i+j=5} S^{(i,4)} S^{(j,3)} = O(2d)^{-5/2} \quad (8.59)$$

for these contributions, which thus belong to the error term. This completes the proof. \square

Acknowledgements

The work of YMM was supported in part by CONACYT of Mexico. The work of GS was supported in part by NSERC of Canada.

References

- [1] G. Aleksandrowicz and G. Barequet. The growth rate of high-dimensional tree polycubes. Preprint, (2012).
- [2] R. Barequet, G. Barequet, and G. Rote. Formulae and growth rates of high-dimensional polycubes. *Combinatorica*, **30**:257–275, (2010). MR2728490
- [3] C. Borgs, J.T. Chayes, R. van der Hofstad, and G. Slade. Mean-field lattice trees. *Ann. Combinatorics*, **3**:205–221, (1999). MR1772346
- [4] N. Clisby, R. Liang, and G. Slade. Self-avoiding walk enumeration via the lace expansion. *J. Phys. A: Math. Theor.*, **40**:10973–11017, (2007). MR2396212
- [5] E. Derbez and G. Slade. Lattice trees and super-Brownian motion. *Canad. Math. Bull.*, **40**:19–38, (1997). MR1443722
- [6] E. Derbez and G. Slade. The scaling limit of lattice trees in high dimensions. *Commun. Math. Phys.*, **193**:69–104, (1998). MR1620301
- [7] M.E. Fisher and D.S. Gaunt. Ising model and self-avoiding walks on hypercubical lattices and “high-density” expansions. *Phys. Rev.*, **133**:A224–A239, (1964).
- [8] D.S. Gaunt and P.J. Peard. $1/d$ -expansions for the free energy of weakly embedded site animal models of branched polymers. *J. Phys. A: Math. Gen.*, **33**:7515–7539, (2000). MR1802107

- [9] D.S. Gaunt, P.J. Peard, C.E. Soteris, and S.G. Whittington. Relationships between growth constants for animals and trees. *J. Phys. A: Math. Gen.*, **27**:7343–7351, (1994). MR1310273
- [10] D.S. Gaunt and H. Ruskin. Bond percolation processes in d dimensions. *J. Phys. A: Math. Gen.*, **11**:1369–1380, (1978).
- [11] B.T. Graham. Borel-type bounds for the self-avoiding walk connective constant. *J. Phys. A: Math. Theor.*, **43**:235001, (2010). MR2646672
- [12] T. Hara. Decay of correlations in nearest-neighbor self-avoiding walk, percolation, lattice trees and animals. *Ann. Probab.*, **36**:530–593, (2008). MR2393990
- [13] T. Hara and G. Slade. On the upper critical dimension of lattice trees and lattice animals. *J. Stat. Phys.*, **59**:1469–1510, (1990). MR1063208
- [14] T. Hara and G. Slade. The number and size of branched polymers in high dimensions. *J. Stat. Phys.*, **67**:1009–1038, (1992). MR1170084
- [15] T. Hara and G. Slade. The self-avoiding-walk and percolation critical points in high dimensions. *Combin. Probab. Comput.*, **4**:197–215, (1995). MR1356575
- [16] A.B. Harris. Renormalized $(1/\sigma)$ expansion for lattice animals and localization. *Phys. Rev. B*, **26**:337–366, (1982). MR0668821
- [17] R. van der Hofstad and A. Sakai. Critical points for spread-out self-avoiding walk, percolation and the contact process. *Probab. Theory Related Fields*, **132**:438–470, (2005). MR2197108
- [18] R. van der Hofstad and G. Slade. Asymptotic expansions in n^{-1} for percolation critical values on the n -cube and Z^n . *Random Struct. Alg.*, **27**:331–357, (2005). MR2162602
- [19] R. van der Hofstad and G. Slade. Expansion in n^{-1} for percolation critical values on the n -cube and Z^n : the first three terms. *Combin. Probab. Comput.*, **15**:695–713, (2006). MR2248322
- [20] M. Holmes. Convergence of lattice trees to super-Brownian motion above the critical dimension. *Electr. J. Probab.*, **13**:671–755, (2008). MR2399294
- [21] E. J. Janse van Rensburg. *The Statistical Mechanics of Interacting Walks, Polygons, Animals and Vesicles*. Oxford University Press, Oxford, (2000). MR1858028
- [22] D.A. Klarner. Cell growth problems. *Canad. J. Math.*, **19**:851–863, (1967). MR0214489
- [23] D.J. Klein. Rigorous results for branched polymer models with excluded volume. *J. Chem. Phys.*, **75**:5186–5189, (1981). *J. Chem. Phys.*, **75**:5186–5189, (1981).
- [24] N. Madras. A pattern theorem for lattice clusters. *Ann. Combinatorics*, **3**:357–384, (1999). MR1772355
- [25] Y. Mejía Miranda. *The critical points of lattice trees and lattice animals in high dimensions*. PhD thesis, University of British Columbia, (2012).

- [26] Y. Mejía Miranda and G. Slade. The growth constants of lattice trees and lattice animals in high dimensions. *Elect. Comm. Probab.*, **16**:129–136, (2011). MR2775351
- [27] P.J. Peard and D.S. Gaunt. $1/d$ -expansions for the free energy of lattice animal models of a self-interacting branched polymer. *J. Phys. A: Math. Gen.*, **28**:6109–6124, (1995). MR1364786
- [28] M.D. Penrose. On the spread-out limit for bond and continuum percolation. *Ann. Appl. Probab.*, **3**:253–276, (1992). MR1202526
- [29] M.D. Penrose. Self-avoiding walks and trees in spread-out lattices. *J. Stat. Phys.*, **77**:3–15, (1994). MR1300525
- [30] G. Slade. *The Lace Expansion and its Applications*. Springer, Berlin, (2006). Lecture Notes in Mathematics Vol. 1879. Ecole d’Eté de Probabilités de Saint–Flour XXXIV–2004. MR2239599